Complex-Distance Potential Theory, Wave Equations, and Physical Wavelets

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Abstract

Potential theory in $\mathbb{R}^n$ is extended to $\mathbb{C}^n$ by analytically continuing the Euclidean distance function. The extended Newtonian potential $\phi(\boldsymbol{z})$ is generated by a *(non-holomorphic)* source distribution $\tilde{\delta}(\boldsymbol{z})$ extending the usual point source $\delta(\boldsymbol{x})$. With Minkowski space $\mathbb{R}^{n,1}$ embedded in $\mathbb{C}^{n+1}$, the Laplacian Δ_{n+1} restricts to the wave operator $\Box_{n,1}$ in $\mathbb{R}^{n,1}$. We show that $\tilde{\delta}(\boldsymbol{z})$ acts as a *propagator* generating solutions of the wave equation from their initial values, where the Cauchy data need *not* be assumed analytic. This generalizes an old result by Garabedian, who established a connection between solutions of the boudary-value problem for Δ_{n+1} and the initial-value problem for $\Box_{n,1}$ provided the boundary data extends holomorphically to the initial data. We relate these results to the *physical wavelets* introduced previously. In the context of Clifford analysis, our methods can be used to extend the Borel-Pompeiu formula from $\mathbb{R}^{n+1}$ to $\mathbb{C}^{n+1}$, where its restriction to Minkowski space $\mathbb{R}^{n,1}$ provides solutions for time-dependent Maxwell and Dirac equations.

1 Holomorphic Potentials and their Sources

Potential theory in $\mathbb{R}^n$ is based on the Euclidean distance

$$r(\boldsymbol{x}) = |\boldsymbol{x}| = \sqrt{\boldsymbol{x}^2}, \qquad \boldsymbol{x} \in \mathbb{R}^n,\ \boldsymbol{x}^2 \equiv \boldsymbol{x} \cdot \boldsymbol{x}.$$

Fundamental solutions for the Laplacian

$$\Delta\phi(\boldsymbol{x}) = \delta(\boldsymbol{x})$$

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are

$$\phi(\boldsymbol{x}) = \frac{1}{2\pi}\ln r, \qquad n = 2$$
$$\phi(\boldsymbol{x}) = \frac{r^{2-n}}{(2-n)\omega_n}, \quad n \geq 3, \quad \text{where} \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the area of the unit sphere in $\mathbb{R}^n$. To extend this to $\mathbb{C}^n$, let

$$\boldsymbol{z} = \boldsymbol{x} - i\boldsymbol{y} \in \mathbb{C}^n, \qquad |\boldsymbol{x}| \equiv r, \quad |\boldsymbol{y}| \equiv a,$$

and define the *complex distance function*

$$\gamma(\boldsymbol{z}) = \sqrt{\boldsymbol{z}^2} \equiv \sqrt{r^2 - a^2 - 2i\boldsymbol{x}\cdot\boldsymbol{y}}\,.$$

The branch points of γ form the *null cone* in $\mathbb{C}^n$,

$$\mathcal{N} \equiv \{\boldsymbol{z} \in \mathbb{C}^n : \ \boldsymbol{z}^2 = 0\}.$$

Fixing $\boldsymbol{y} \neq \boldsymbol{0}$, this gives the following set in $\mathbb{R}^n$:

$$S(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^n : \ r = a, \ \boldsymbol{x}\cdot\boldsymbol{y} = 0\},$$

a sphere of radius a in $\boldsymbol{y}^\perp \approx \mathbb{R}^{n-1}$. We may regard $S(\boldsymbol{y})$ as the set of points in $\mathbb{R}^n$ whose complex distance from the *imaginary source point* $i\boldsymbol{y}$ vanishes. (In the context of spacetime, we will interpret $|\boldsymbol{y}| \equiv t$ as *time*, , $\boldsymbol{y}^\perp \approx \mathbb{R}^{n-1}$ as *space*, and $S(\boldsymbol{y}) = \{\boldsymbol{x} \in \boldsymbol{y}^\perp : |\boldsymbol{x}| = t\}$ as a section of the light cone.)

To make γ single-valued, choose a *branch cut* so that

$$\boldsymbol{y} \to 0 \ \Rightarrow \ \gamma(\boldsymbol{x} - i\boldsymbol{y}) \to r(\boldsymbol{x}) \geq 0. \tag{$\mathcal{B}$}$$

The simplest such choice is

$$\text{Re}\ \gamma \geq 0.$$

Then $\text{Im}\ \gamma$ is discontinuous across the *disk*

$$D(\boldsymbol{y}) = \{\boldsymbol{x} : \ r \leq a, \ \boldsymbol{x}\cdot\boldsymbol{y} = 0\}, \quad \partial D(\boldsymbol{y}) = S(\boldsymbol{y}).$$

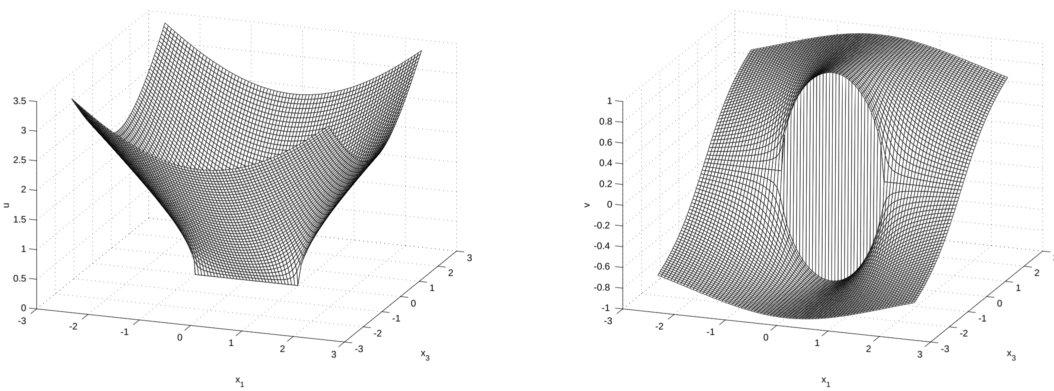

Figure 1. Plots of Re γ and Im γ with $n = 2$ and $\boldsymbol{y} = (0, -1)$. $D(\boldsymbol{y})$ is now the interval $[-1, 1]$ along the x_1 axis. The graph of Re $\gamma(\boldsymbol{x})$ is a pinched cone, and the discontinuity of Im $\gamma(\boldsymbol{x})$ across $D(\boldsymbol{y})$ is circular, being greatest at the origin.

A *general* branch cut satisfying ($\mathcal{B}$) is obtained by continuously deforming $D(\boldsymbol{y})$ to any *membrane* M bounded by $S(\boldsymbol{y})$:

$$\partial M = S(\boldsymbol{y}).$$

Define the *analytic potential* in $\mathbb{C}^n$ as

$$\phi(\boldsymbol{z}) = \frac{1}{2\pi} \ln \gamma, \qquad n = 2$$
$$\phi(\boldsymbol{z}) = \frac{\gamma^{2-n}}{(2-n)\omega_n}, \qquad n \geq 3.$$

For *even* $n \geq 4$, ϕ is analytic in $\{\boldsymbol{x} - i\boldsymbol{y} : \boldsymbol{x} \notin S(\boldsymbol{y})\}$. For all other $n \geq 2$, ϕ is analytic in $\{\boldsymbol{x} - i\boldsymbol{y} : \boldsymbol{x} \notin D(\boldsymbol{y})\}$. We define the *source distribution* $\tilde{\delta}(\boldsymbol{z})$ of ϕ by

$$\boxed{\tilde{\delta}(\boldsymbol{z}) \equiv \Delta_{\boldsymbol{x}} \, \phi(\boldsymbol{z})}$$

where $\Delta_{\boldsymbol{x}}$ is the distributional Laplacian with respect to $\boldsymbol{x}$ at constant $\boldsymbol{y}$. Note that $\phi(\boldsymbol{x}+i\boldsymbol{y})$ is harmonic in $\boldsymbol{x}$ wherever it is analytic. This leads to the following result.

Theorem 1 [K00] *Fix* $\boldsymbol{y} \neq \mathbf{0}$*. Then*

- $\tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y})$ *is a distribution in* $\boldsymbol{x}$ *with support*

$$\operatorname{supp} \tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y}) = S(\boldsymbol{y}) \qquad \textit{for even } n \geq 4.$$
$$\operatorname{supp} \tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y}) = D(\boldsymbol{y}) \qquad \textit{for } n = 2 \textit{ and odd } n \geq 3.$$

(In the context of spacetime, this will be related to Huygens' principle.)

- $\boldsymbol{y} \to \mathbf{0} \;\Rightarrow\; \tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y}) \to \delta(\boldsymbol{x})$, *and we have the following* moments:
  - *Monopole:* $$Q(\boldsymbol{y}) \equiv \int_{\mathbb{R}^n} \tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y})\, d\boldsymbol{x} = 1$$
  - *Dipole:* $$\boldsymbol{P}(\boldsymbol{y}) \equiv \int_{\mathbb{R}^n} \boldsymbol{x}\, \tilde{\delta}(\boldsymbol{x} - i\boldsymbol{y})\, d\boldsymbol{x} = i\boldsymbol{y}$$
  - *Centroid:* $$\boldsymbol{C}(\boldsymbol{z}) \equiv \int_{\mathbb{R}^n} \boldsymbol{x}\, \tilde{\delta}(\boldsymbol{x} - \boldsymbol{z})\, d\boldsymbol{x} = \boldsymbol{z} \quad \forall \boldsymbol{z} \in \mathbb{C}^n.$$

- *In particular, for* $n = 3$ *and a test function* $f : \mathbb{R}^3 \to \mathbb{R}$*, we have*

$$\langle\, \tilde{\delta}\,, f \,\rangle = f(S) + aL_1(f) + iaL_2(f) \quad \textit{where}$$

$f(S)$ is the *mean* of f over the circle $S(\boldsymbol{y})$

$L_1(f)$ is a real *simple (charge) layer* on the disk $D(\boldsymbol{y})$

$L_2(f)$ is a real *double (polarization) layer* on $D(\boldsymbol{y})$.

The expressions for $L_1(f), L_2(f)$, as well as the complete expression for $\langle\, \tilde{\delta}\,, f \,\rangle$ in the case of general n, are given in [K00].

2 How $\tilde{\delta}(\boldsymbol{z})$ Generates Beams in $\mathbb{R}^3$

For $n = 3$,

$$\phi_k(\boldsymbol{z}) \equiv -\frac{e^{ik\gamma(\boldsymbol{z})}}{4\pi\gamma(\boldsymbol{z})}, \qquad k \geq 0$$

is an extended fundamental solution for the Helmholtz equation:

$$\Delta_{\boldsymbol{x}} \phi_k(\boldsymbol{z}) + k^2 \phi_k(\boldsymbol{z}) = \tilde{\delta}(\boldsymbol{z}).$$

(Actually, $\phi_k(\boldsymbol{z})$ must be *regularized* as in [K00] for this to make mathematical sense.) In the *far zone*, defined by $r \gg a$, we have

$$\gamma(\boldsymbol{x} - i\boldsymbol{y}) \equiv \sqrt{r^2 - a^2 - 2ira\cos\theta} \approx r - ia\cos\theta, \quad \text{where} \quad \boldsymbol{x} \cdot \boldsymbol{y} \equiv ra\cos\theta,$$

and

$$\phi_k(\boldsymbol{x} - i\boldsymbol{y}) \approx -\frac{e^{ikr}\, e^{ka\cos\theta}}{4\pi r}\,.$$

> Under the extension $r(\boldsymbol{x}) \to \gamma(\boldsymbol{x} - i\boldsymbol{y})$, interpreted as a displacement of the point source from $\mathbf{0}$ to $i\boldsymbol{y}$, the spherically symmetric fundamental solution $-e^{ikr}/4\pi r$ of the Helmoltz equation becomes a *well-directed beam* emitted by $D(\boldsymbol{y})$ in the positive $\boldsymbol{y}$ direction.

Applications to radar and communications, where $D(\boldsymbol{y})$ or its deformations M are interpreted as *antenna dishes,* are proposed in [K94a] and [K00a].

3 Connection to Spacetime and Wave Equations

Denote points in $\mathbb{R}^{n+1}$ (regarded as *Euclidean spacetime*) by

$$\boldsymbol{X} = (\boldsymbol{x}, s), \qquad \boldsymbol{x} \in \mathbb{R}^n,\ s \in \mathbb{R}, \qquad \boldsymbol{X}^2 = r^2 + s^2.$$

Consider the boundary-value problem for Laplace's equation in $\mathbb{R}^{n+1}$,

$$\Delta_{\boldsymbol{X}}\, u(\boldsymbol{X}) = (\Delta_{\boldsymbol{x}} + \partial_s^2) u(\boldsymbol{x}, s) = 0,$$

and the the initial-value problem for wave equation in the *Minkowski spacetime* $\mathbb{R}^{n,1}$,

$$\Box v(\boldsymbol{x}, t) \equiv (\Delta_{\boldsymbol{x}} - \partial_t^2) v(\boldsymbol{x}, t) = 0.$$

These are related purely *formally* by

$$v(\boldsymbol{x}, t) = u(\boldsymbol{x}, it).$$

But this makes sense only if $u(\boldsymbol{x}, s)$ is analytic in s. Garabedian [G64] established a connection between u and v, *assuming that the boundary values of u are analytic in s.*

The relation to the present work is as follows. Let

$$\begin{aligned} \boldsymbol{z} &= \boldsymbol{x} - i\boldsymbol{y} \in \mathbb{C}^n, \quad |\boldsymbol{x}| = r,\ |\boldsymbol{y}| = a \\ \boldsymbol{X} &= (\boldsymbol{x}, s) \in \mathbb{R}^{n+1}, \quad \boldsymbol{Y} = (\boldsymbol{y}, -t) \in \mathbb{R}^{n+1} \\ \boldsymbol{Z} &= \boldsymbol{X} - i\boldsymbol{Y} = (\boldsymbol{z}, s + it) \in \mathbb{C}^{n+1}. \end{aligned}$$

The complex distance Γ in $\mathbb{C}^{n+1}$ is defined by

$$\Gamma^2 \equiv \boldsymbol{Z}^2 = \boldsymbol{X}^2 - \boldsymbol{Y}^2 - 2i\boldsymbol{X} \cdot \boldsymbol{Y}, \quad \operatorname{Re} \Gamma \geq 0.$$

This reduces to the *Euclidean* metric in $\mathbb{R}^{n+1}$

$$\Gamma_E^2 = r^2 + s^2 \quad \text{when} \quad \boldsymbol{Y} \to \mathbf{0},$$

and to the *Lorentzian* metric in $\mathbb{R}^{n,1}$

$$\Gamma_L^2 = r^2 - t^2 \quad \text{when} \quad s \to 0,\ \boldsymbol{y} \to \mathbf{0}.$$

Note that

$$\Gamma_E = 0 \ \Rightarrow \ \boldsymbol{X} = \boldsymbol{0} \qquad \text{(point source in elliptic case)}$$
$$\Gamma_L = 0 \ \Rightarrow \ r = |t| \qquad \text{(light cone in hyperbolic case).}$$

If there are no prefered directions in the Euclidean world $\mathbb{R}^{n+1}$, then we may choose a coordinate system where

$$\boldsymbol{Y} = (\boldsymbol{0}, -t), \ \Rightarrow \ \boldsymbol{Z} = (\boldsymbol{x}, s + it).$$

Recall the definition of $\tilde{\delta}$, now with $n \to n+1$:

$$\tilde{\delta}(\boldsymbol{Z}) \equiv \Delta_{\boldsymbol{X}} \Phi(\boldsymbol{Z}), \quad \Phi(\boldsymbol{Z}) = \frac{\Gamma(\boldsymbol{Z})^{1-n}}{(1-n)\omega_{n+1}}.$$

For odd $n \geq 3$, $\tilde{\delta}(\boldsymbol{X} - i\boldsymbol{Y})$ is supported on $\Gamma = 0$:

$$\boldsymbol{X} \cdot \boldsymbol{Y} = 0 \ \Rightarrow \ s = 0$$
$$\boldsymbol{X}^2 = \boldsymbol{Y}^2 \ \Rightarrow \ r = |t|,$$

which is the *light cone in real* spacetime. For all other n, $\tilde{\delta}(\boldsymbol{X} - i\boldsymbol{Y})$ is supported on the solid *causal cone*

$$\boldsymbol{X} \cdot \boldsymbol{Y} = 0 \ \Rightarrow \ s = 0$$
$$\boldsymbol{X}^2 \leq \boldsymbol{Y}^2 \ \Rightarrow \ r \leq |t|.$$

For a sufficeintly smooth test function f ($f \in C^k(\mathbb{R}^{n+1})$, where k increases with n), define the convolution

$$\tilde{f}(\boldsymbol{Z}) = \int_{\mathbb{R}^{n+1}} \tilde{\delta}(\boldsymbol{Z} - \boldsymbol{X}')\, f(\boldsymbol{X}')\, d\boldsymbol{X}', \qquad \boldsymbol{Z} \in \mathbb{C}^{n+1}.$$

Theorem 2 *$\tilde{f}(\boldsymbol{x}, s+it)$ solves the following initial-value problem for the wave equation in $\mathbb{R}^{n,1}$:*

$$\lim_{t\to 0} \tilde{f}(\boldsymbol{x}, s+it) = f(\boldsymbol{x}, s) \tag{1}$$

$$\lim_{t\to 0} \partial_t \tilde{f}(\boldsymbol{x}, s+it) = i\partial_s f(\boldsymbol{x}, s) \tag{2}$$

$$(\Delta_{\boldsymbol{x}} - \partial_t^2)\tilde{f}(\boldsymbol{x}, s+it) = 0. \tag{3}$$

Sketch of proof: For odd $n \geq 3$, $\tilde{\delta}(\boldsymbol{Z} - \boldsymbol{X}')$ is supported on a sphere of radius $|t|$ in $\boldsymbol{Y}^{\perp}$. Thus $\tilde{f}(\boldsymbol{Z})$ is a combination of *spherical means* of f and its derivatives. This turns out to be precisely the expression for the solution of the above initial-value problem, as expressed in terms of spherical means [J55]. For all other values of n, we apply a variant of Hadamard's *method of descent*. See [K00] for details.

$\tilde{\delta}$ is an *extended source* (a sphere or disk) in the elliptic case and a *propagator* in the hyperbolic case. *Huygens' principle* is a simple consequence of the fact that for odd $n \geq 3$, $\tilde{\delta}$ is supported on the light cone and so the solution $\tilde{f}$ depends on the initial values only there. For all other $n \geq 1$, $\tilde{\delta}$ is supported on the solid cone $r \leq |t|$ in $\mathbb{R}^{n,1}$, and we merely have *causality.*

The initial condition (1) states that $\tilde{f}$ is an *extension* of f, which follows from

$$\boldsymbol{Y} \to \boldsymbol{0} \;\Rightarrow\; \tilde{\delta}(\boldsymbol{X} - i\boldsymbol{Y}) \to \delta(\boldsymbol{X})$$

while (2) is the *Cauchy-Riemann equation* for $\tilde{f}(\boldsymbol{x}, s+it)$ at $t = 0$. But we need *not* assume that the boundary data $f(\boldsymbol{x}, s)$ is analytic in s, since the CR equation need not hold for $t \neq 0$! This distinguishes our results from similar results in the literature, where analyticity in s of the boundary data is required; see Garabedian [G64, pp. 191–202] and Ryan [R90, R90a].

4 Pulsed-Beam Wavelets

We now specialize to the wave equation in $\mathbb{R}^{3,1}$. Consider an observer with *real spacetime coordinates*

$$\boldsymbol{Z}_1 = (\boldsymbol{x}, it) \in \mathbb{R}^{3,1}$$

and a "point source" with *imaginary spacetime coordinates*

$$\boldsymbol{Z}_2 = i(\boldsymbol{y}, is) \in i\,\mathbb{R}^{3,1} = \mathbb{R}^{1,3}.$$

The complex relative position vector is

$$\boldsymbol{Z} = \boldsymbol{Z}_1 - \boldsymbol{Z}_2 = (\boldsymbol{x} - i\boldsymbol{y}, s+it) \equiv (\boldsymbol{z}, s+it).$$

The complex distance $\Gamma = \sqrt{\boldsymbol{Z}^2}$ in $\mathbb{C}^4$ is related to the complex distance $\gamma = \sqrt{\boldsymbol{z}^2}$ in $\mathbb{C}^3$ by

$$\Gamma^2 = \gamma^2 + (s+it)^2 = \gamma^2 - (t-is)^2.$$

The analytic potential in $\mathbb{C}^4$ therefore splits into two parts:

$$\Phi(\boldsymbol{Z}) \equiv -\frac{1}{4\pi^2\,\Gamma^2} = \frac{1}{4\pi^2(t-is-\gamma)(t-is+\gamma)} = \Phi_+(\boldsymbol{Z}) - \Phi_-(\boldsymbol{Z}),$$

where

$$\Phi_\pm(\boldsymbol{z}, s+it) = \frac{1}{8\pi^2\big(t - is \mp \gamma(\boldsymbol{z})\big)\gamma(\boldsymbol{z})}\,. \tag{$\mathcal{W}$}$$

In the far zone $r \gg a$, where $\gamma(\boldsymbol{x} - i\boldsymbol{y}) \approx r - ia\cos\theta$, ($\mathcal{W}$) gives the *far fields*

$$\Phi_\pm(\boldsymbol{z}, s+it) \approx \frac{1}{8\pi^2\,r} \cdot \frac{1}{t \mp r - i(s \mp a\cos\theta)}. \tag{$\mathcal{F}$}$$

To make $\Phi_\pm$ nonsingular, it is necessary and sufficient to assume

$$|s| > |\boldsymbol{y}| = a,$$

which means that $(\boldsymbol{y}, is)$ belongs to the *future cone* $(s > a)$ or *past cone* $(s < -a)$, hence $\boldsymbol{Z}$ belongs to the *future tube* $\mathcal{T}_+$ or *past tube* $\mathcal{T}_-$ [K94a]. (Note that $-a \le \operatorname{Im}\gamma(\boldsymbol{z}) \le a$ for *all* $\boldsymbol{z} \in \mathbb{C}^n$, not only in the far zone [K00].) For definiteness we assume $s > a$. Then $(\mathcal{F})$ shows that

- Φ_+ is *causal,* peaking at $t = r$ and $\theta = 0$
- Φ_- is *anticausal,* peaking at $t = -r$ and $\theta = \pi$
- $\Phi_\pm$ are *pulsed beams* of (θ-dependent) duration

  $$\tau(\theta) = s \mp a\cos\theta,$$

  emitted and absorbed, respectively, along $\pm\boldsymbol{y}$.
- As $s \to a^+$, $\Phi_\pm$ become more and more *focused.*

> $\Phi_+(\boldsymbol{z}, s+it)$ is a causal pulsed beam emitted by $D(\boldsymbol{y})$ along $\boldsymbol{y}$, and $\Phi_-(\boldsymbol{z}, s+it)$ is an anticausal pulsed beam absorbed by $D(\boldsymbol{y})$ along $-\boldsymbol{y}$. *The direction, duration and directivity of these beams are determined by the imaginary spacetime coordinates* $i(\boldsymbol{y}, is)$ *of the source.*

Solutions of the form $(\mathcal{W})$ are known in the engineering literature as *complex-source pulsed beams* [HF89, HLK99]. The idea was introduced independently under the name *physical (acoustic and electromagnetic) wavelets* in [K92, K94, K94a], and its relation to the complex-distance potential theory as formulated here sheds some light on the possibility of realizing such wavelets by constructing their physical sources.

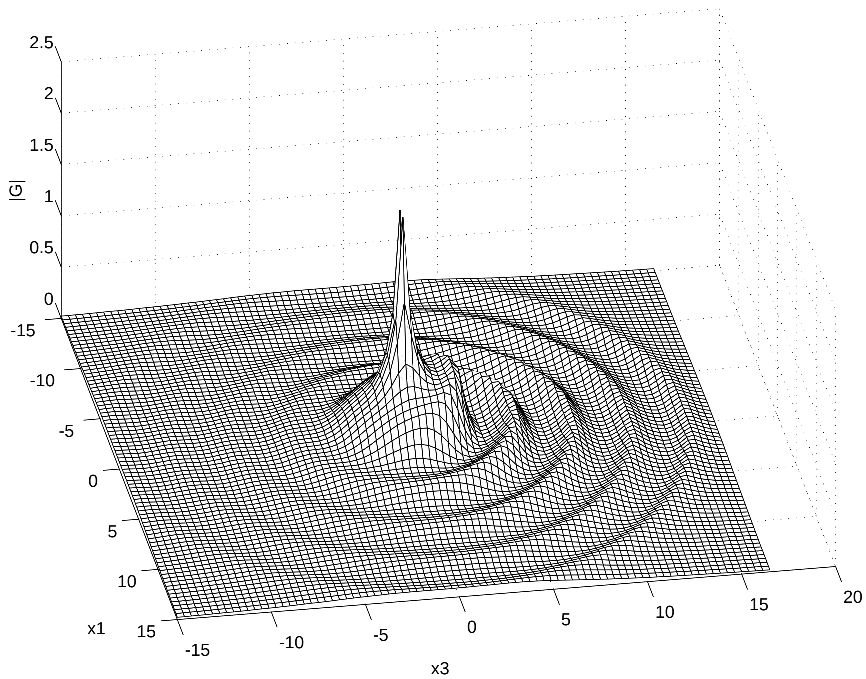

Figure 2. *Time-lapse plot* of $|\Phi_+(\boldsymbol{x} - i\boldsymbol{y}, s + it)|$ as a function of (x_1, x_3) with $x_2 = 0$, superimposed at times t =3, 6, 9, 12, 15. We have fixed $\boldsymbol{y} = (0, 0, .5)$ and $s = 1$. The suppression of x_2 is no loss because of the axial symmetry about $\boldsymbol{y}$. The imaginary "point source" coordinates are $i(0, 0, .5, i)$, hence $s - a = 1/2$ and the pulsed beam has some directivity along the x_3 axis.

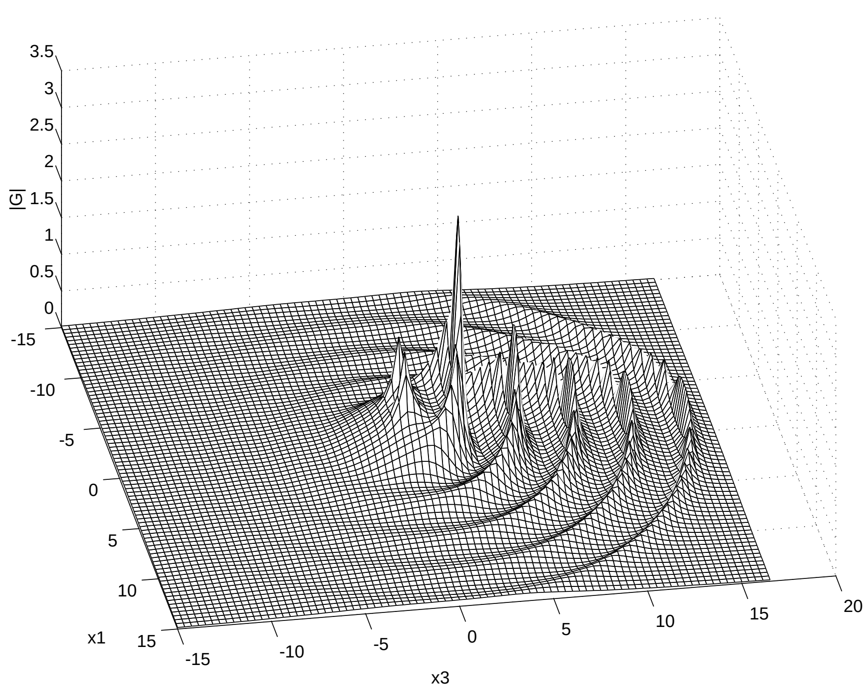

Figure 3. Same as Figure 2 but with $\boldsymbol{y} = (0, 0, .9)$, giving greater directivity $(s - a = 1/10)$.

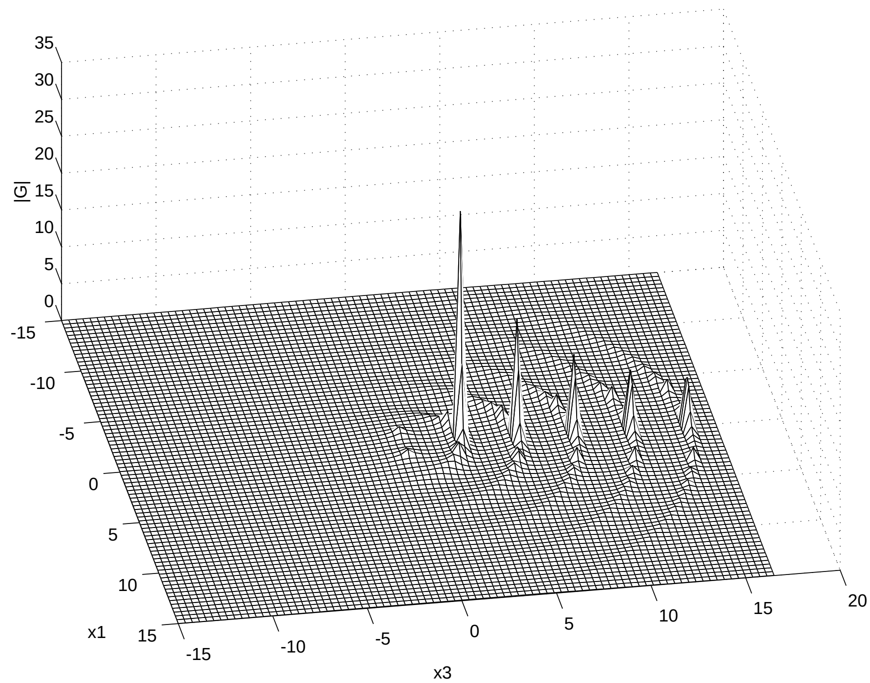

Figure 4. Same as Figure 2 but with $\boldsymbol{y} = (0, 0, .99)$, giving still greater directivity.

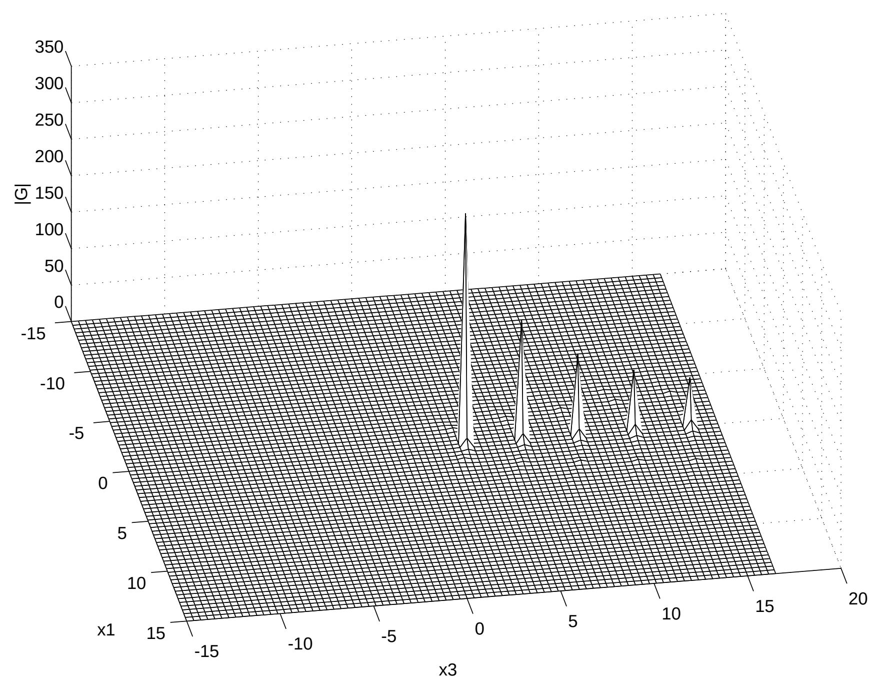

Figure 5. Same as Figure 4, but with $\boldsymbol{y} = (0, 0, .999)$, further increasing the directivity. The beam is highly focused along the propagation axis.

5 Extension to Clifford Analysis, Maxwell's Equations

Our results readily extend to Clifford analsysis, where they can be applied in particular to the time-dependent Maxwell and Dirac equations [K00]. Clifford analysis [BDS82, R96, GS97] is a generalization to $\mathbb{R}^n$ of one-dimensional complex analysis (alternative and inequivalent to the usual complex analysis in several variables) that is proving to be a powerful tool in physics [H66, KS96, MM98, O98, B99]. One of the major difficulties in this field is that standard results, such as the Borel-Pompeiu formula [GS97] (a multidimensional generalization of the *inhomogeneous Cauchy formula* [T75]), apply only to elliptic equations, hence they exclude physically important settings such as the time-dependent (hyperbolic) Maxwell and Dirac equations. Although some progress has been made in this direction through analytic continuation [R90, R90a], this depends, like Garabedian's work [G64], on the assumption that bounday and initial data are related by analyticity. Since our method is independent of this requirement, it provides a more general connection between the elliptic Dirac and Maxwell equations, formulated in terms of *real* Clifford analysis, and their hyperbolic versions, obtained by its extension based on complex distance.

In particular, pulsed-beam wavelets can be constructed for Maxwell's equations closely related to the *electromagnetic wavelets* introduced in [K94, K94a]. There is an unexpected connection to the electromagnetic fields of Kerr-Newman spinning, charged black holes [N73]. In our notation, Newman [N73] defines the electromagnetic field of the *linearized* Maxwell-Einstein system by extending the Coulomb field in flat space to $\mathbb{C}^3$,

$$\boldsymbol{F}(\boldsymbol{z}) = -\boldsymbol{\nabla}\frac{1}{4\pi\gamma(\boldsymbol{z})} = \frac{\boldsymbol{z}}{4\pi\gamma(\boldsymbol{z})^3}\,, \quad \boldsymbol{z} = \boldsymbol{x} - i\boldsymbol{y} \in \mathbb{C}^3 \tag{4}$$

and identifying the real and imaginary parts of $\boldsymbol{F}$ as electric and magnetic fields, respectively:

$$\boldsymbol{F}(\boldsymbol{z}) \equiv \boldsymbol{E}(\boldsymbol{z}) - i\boldsymbol{B}(\boldsymbol{z}).$$

Newman ignores the sources, considering $\boldsymbol{F}$ as a solution of the *homogeneous* Maxwell equations outside its singularity set. But in the context of Clifford analysis, (4) is an extension to $\mathbb{C}^3$ of the *Cauchy kernel*

$$\boldsymbol{C}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{4\pi|\boldsymbol{x}|^3}\,, \quad \boldsymbol{x} \in \mathbb{R}^3,$$

a fundamental solution of the (elliptic) Dirac operator in $\mathbb{R}^3$. I have computed the distributional charge-current density of $\boldsymbol{F}$ and showed that its source is a charged disk $D(\boldsymbol{y})$ *spinning rigidly at the critical angular velocity,* so that its rim moves at the speed of light [K01]. This is consistent with the interpretation of the Kerr-Newman solution as a apinning, charged black hole.

The results described here will be elaborated, with a view to electrodynamics, relativistic quantum mechanics and quantum fields, in a forthcoming book [K02].

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