A SAMPLING THEOREM FOR SIGNALS IN THE JOINT TIME-FREQUENCY DOMAIN

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ABSTRACT

Certain analog systems, such as the ear, have outputs which are time-dependent spectra. Such a system can be modeled as the Fourier transform of a windowed version of the input signal, sometimes called a short-term Fourier transform. It is shown that the input signal can be fully recovered by sampling the output at time-intervals T and frequency-intervals F, where $TF \leq 1$.

A short-term Fourier transform of a signal $x(t)$ may be defined as follows: For each time, s, choose a window $h_s(t)$ which specifies how much emphasis is given to the signal at time t in computing the output at time s. Then the short-term Fourier transform of $x(t)$ is defined as

$$X(f,s) = \int_{-\infty}^{\infty} e^{-j2\pi ft}\, h_s(t)x(t)dt. \tag{1}$$

It is usually assumed that the system is time-invariant, i.e. $h_s(t) = h_0(t-s)$, but this is not necessary for our purposes. We do assume that the system is causal ($h_s(t) = 0$ for $t > s$) and has a finite memory τ independent of s ($h_s(t) = 0$ for $t \leq s-\tau$). Choose a time-interval $\Delta s =$ and a frequency-interval $\Delta f = F$, and consider the samples

$$X_{kn} = X(kF,nT) \tag{2}$$

of $X(f,s)$ for integers k and n. We wish to know under what conditions $x(t)$ can be recovered from these samples. Suppose $F \leq \frac{1}{\tau}$. Then the windowed signal $h_{nT}(t)x(t)$ vanishes outside the interval $nT - \frac{1}{F} < t \leq n$ hence can be expanded in a Fourier series

$$h_{nT}(t)x(t) = \sum_k C_{kn}\, e^{j2\pi kFt} \tag{3}$$

on that interval, with

$$C_{kn} = F \int_{s-\frac{1}{F}}^{s} e^{-j2\pi kFt} h_{nT}(t)x(t)dt$$

$$= FX_{kn}. \tag{4}$$

To make the expansion (3) valid for all t (i.e., eliminate the periodic repetitions on the right-hand side), multiply both sides by $h_{nT}(t)^*$:

$$|h_{nT}(t)|^2 x(t) = F\, h_{nT}(t)^* \sum_k X_{kn}\, e^{j2\pi kFt}. \tag{5}$$

Thus

$$\left[\sum_n |h_{nT}(t)|^2\right] x(t) = F \sum_n \sum_k X_{kn}\, h_{nT}(t)^*\, e^{j2\pi kFt}. \tag{6}$$

To recover x(t), we must now assume that the function

$$g(t) = \left[\sum_n |h_{nT}(t)|^2\right]^{-1} \tag{7}$$

is bounded for all t. This implies that $T \le \tau$ (for otherwise

$\sum_n |h_{nT}(t)|^2 = 0$ for $nT < t \le nT + T-\tau$). If g(t) is bounded, we may

recover x(t):

$$x(t) = F\, g(t) \sum_n \sum_k X_{kn}\, h_{nT}(t)^* \, e^{j2\pi kFt}. \tag{8}$$

To summarize, the conditions necessary for recovery are

$$T \le \tau \le \frac{1}{F}\ , \tag{9}$$

from which it follows that $FT \le 1$, i.e., we have at least one sample per unit area in the time-frequency plane. We wish to note the following:

1. All sums over n are finite, since $h_{nT}(t) = 0$ unless $nT-\tau < t \le nT$.
2. In principle it is unnecessary to assume that x(t) is band-limited, as in the Nyquist sampling theorem. In fact, if the windows $h_{nT}(t)$ are reasonable (e.g., piecewise continuous), it is only required that x(t) be square-integrable on each interval $nT-\tau \le t \le nT$. Thus x(t) need only have finite average power.
3. In practice, the formula (8) cannot be used as it stands because it would require an infinite number of samples $X_{kn}(k=0, \pm 1, \pm 2, \ldots)$ at each time nT. But a good approximation can be expected by truncating the sum at finite k. In fact, this gives something akin

to a band-limited interpolation of x(t), since the Fourier transform of (8) gives

$$X(f) = \sum_n \sum_k X_{kn} L_n(f-kF) \tag{10}$$

where $L_n(f)$ is the Fourier transform of $Fg(t)h_{nT}(t)^*$. Thus if $h_{nT}(t)$ is real and not too rough, $L_n(f)$ behaves as an approximate low-pass filter and is neglible for $|f| >> 1/\tau$. If the sum over k is replaced by a finite sum with $|kF| \leq B$, then (8) requires

$$\frac{2B}{FT} \geq 2B$$

samples per second. In the extreme case (FT = 1), this coincides with the Nyquist rate.

4. If in (6) we sum only over $n_1 \leq n \leq n_2$, then x(t) may still be recovered, but only in the interval $n_1T-\tau < t \leq n_2T$, since

$$\sum_{n=n_1}^{n_2} |h_{nT}(t)|^2 = 0$$

otherwise. The formula (8) is still valid in this interval, where n in both g(t) and (8) runs from n_1 to n_2. (However, g(t) may become unbounded as $t \to n_2T$ and $t \to n_1T-\tau$.) This appears to be an advantage over the Nyquist theorem, where truncation of the time-samples is not so trivial.

5. When applying the Nyquist sampling theorem to a signal which is not band-limited, it is in practice necessary first to pass the signal

through a low-pass filter to avoid aliasing errors. This is unnecessary when using (8).

6. Eq. (8) has an obvious generalization to several independent variables, hence can also be applied to optical signals.

Finally, eq. (8) has the following suggestive interpretation: The signal

$$F\, g(t)\, h_{nT}(t) * e^{j2\pi kFt}$$

may be regarded as a musical note of frequency kF (k-th harmonic of the fundamental frequency F) and duration τ, played at the time $nT-\tau$. Then the collection of samples X_{kn} may be viewed as a <u>score</u> for x(t) in the <u>key</u> F and <u>tempo</u> T, and (8) states that x(t) may be synthesized by "playing" its score!

REFERENCES

1. L.E. Franks, Signal Theory. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1969.