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Quantized Fields in Complex Spacetime

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We extend to quantized fields the phase-space formalism previously developed for one-particle relativistic quantum theory. An integral transform is derived which extends an arbitrary system of fields from real spacetime $\mathbb{R}^4$ to complex spacetime $\mathbb{C}^4$. In the case of free fields, the extension is analytic in the union $\mathscr{T}$ of the forward and backward tubes. The forward tube supports only the positive-frequency part of the field, and the backward tube the negative-frequency part. Observables such as the charge and energy-momentum are obtained by integrating field combinations over phase spaces, which are two-sheeted, six-dimensional submanifolds of $\mathscr{T}$, bounded away from real spacetime $\mathbb{R}^4$. Consequently, the fields and their products are much more regular in phase space than on spacetime. Also, the particles associated with these fields are covariantly extended in space. Propagators carry positive- and negative-frequency components of the fields into the forward and backward tubes, respectively. For the Dirac field, the separation of positive and negative frequencies implies the complete absence of Zitterbewegung. It is furthermore shown that within the axiomatic framework, the interpretation of the complexified spacetime in terms of phase space survives the transition from free to interacting fields. © 1987 Academic Press, Inc.

1. INTRODUCTION

In a previous series of papers we developed a phase-space formulation of first-quantized Klein–Gordon theory [1–4; 6, 7]. At the heart of that work lies the observation that : (a) The positivity of the energy (spectral condition) gives rise to an analytic extension of wave functions from spacetime $\mathbb{R}^4$ to a complex domain, namely the forward tube $\mathscr{T}_+$. (b) This complexified spacetime, familiar in Quantum Field Theory [8], has a physical significance which seems to have been overlooked: it is closely related to the *classical phase space* of the particles of the theory. This insight motivates a radical reformulation of the theory, where inner products (and hence expectation values of observables) are computed by integrating over 6-dimensional submanifolds of $\mathscr{T}_+$ (phase spaces) rather than 3-dimensional submanifolds of $\mathbb{R}^4$ (configuration spaces). Although closely related to the usual formalism, the new one offers some major advantages. The particles it describes are automatically *extended* in space rather than being point-particles. This extension is fully covariant, and increases in size the farther the phase-space is chosen from $\mathbb{R}^4$. This gives a new

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perspective on the localization problem of relativistic particles. Furthermore, the form of the inner product on phase-space yields a natural and covariant probabilistic interpretation, where there is none in the usual theory. Finally, there are indications that gauge theory may simplify considerably in the new framework, with the gauge potentials being derivable from a fiber metric [6, 7].

Thus far, the formalism has been limited to the first-quantized theory because no natural way was seen of extending the quantum fields themselves to complex spacetime. (Some speculations were advanced in [2, 9], but turned out to be inadequate.) In the present paper, we propose a scheme which seems to be both satisfactory and very general. An integral transform (Eq. (9)) is developed which canonically extends any field from real spacetime $\mathbb{R}^4$ to complex spacetime $\mathbb{C}^4$. When applied to a free field, the extension is analytic in the union $\mathscr{T}$ of the forward and backward tubes $\mathscr{T}_+$ and $\mathscr{T}_-$. Furthermore, the positive-frequency (annihilation) part of the field vanishes on $\mathscr{T}_-$ and the negative-frequency (creation) part vanishes on $\mathscr{T}_+$. Observables such as charge and energy-momentum are obtained by integrating field combinations over two-sheeted, six-dimensional phase spaces in $\mathscr{T}$. Because of the separation of positive and negative frequencies, no Wick ordering is necessary to avoid zero-point energies. For the same reason, Zitterbewegung does not occur in the case of the Dirac field. Propagators are derived for the Klein–Gordon and Dirac fields which carry positive frequencies to $\mathscr{T}_+$ and negative frequencies to $\mathscr{T}_-$, thus preserving the separation of the frequencies.

As expected, much less can be said about interacting fields. However, the extension to $\mathbb{C}^4$ still exists, in principle, as does its interpretation in terms of phase space.

The present paper is largely self-contained and can be read independently, although frequent references are made to [2–4]. Our notation and conventions agree with those in [5], with minor exceptions. In Section 2 we develop the formalism for the free Klein–Gordon field and establish its phase-space interpretation. In Section 3 we do the same for the free Dirac field and show that no Zitterbewegung exists in this theory. In Section 4 we prove within the axiomatic framework [8] that the phase-space interpretation of $\mathscr{T}$ is still valid for interacting fields.

2. The Free Klein–Gordon Field

Consider the charged scalar field [5] of mass $m > 0$,

$$\begin{aligned}\phi(x) &= (2\pi)^{-3} \int_{\mathbb{R}^4} d^4p\, \delta(p^2 - m^2)\, e^{-ixp} a(p) \\ &= \int_{\Omega_m} d\tilde{p}\, e^{-ixp} a(p) \\ &= \int_{\Omega_m^+} d\tilde{p} [e^{-ixp} a(p) + e^{ixp} b^\dagger(p)] \qquad (1)\end{aligned}$$

where

$$\Omega_m = \{p \in \mathbb{R}^4 \mid p^2 \equiv p_0^2 - \mathbf{p}^2 = m^2\} \tag{2}$$

is the two-sheeted mass hyperboloid, Ω_m^+ is the positive mass shell, $xp = x_0 p_0 - \mathbf{x} \cdot \mathbf{p}$ and

$$d\tilde{p} = (16\pi^3 \omega)^{-1} d^3p \tag{3}$$

is the Lorentz-invariant measure on Ω_m, with $\omega(\mathbf{p}) = |p_0| = (m^2 + \mathbf{p}^2)^{1/2}$. We use natural units $\hbar = c = 1$. In (1), we have set $b^\dagger(p) \equiv a(-p)$ for p in Ω_m^+, where the dagger denotes hermitian conjugation. The operators $a(p)$ and $b(p)$ satisfy the canonical commutation relations with covariant normalization,

$$[a(p), a^\dagger(q)] = [b(p), b^\dagger(q)] = 16\pi^3\omega\, \delta(\mathbf{p} - \mathbf{q}) \equiv \langle p \mid q \rangle \tag{4}$$

for p and q in Ω_m^+, with all other commutators zero.

We wish to extend $\phi(x)$ to complex spacetime. A simple substitution $x \to z = x - iy \in \mathbb{C}^4$ leads to a divergent integral in (1), since the factor e^{-yp} necessarily grows on some part of Ω_m. To get convergence, we use a device which may at first seem like one of brute force: replace e^{-yp} by $\theta(yp) e^{-yp}$, where θ is the unit step function. Thus, for arbitrary $z = x - iy$ in $\mathbb{C}^4$, define

$$\begin{aligned}\phi(z) &= \int_{\Omega_m} d\tilde{p}\theta(yp)\, e^{-izp} a(p) \\ &= \int_{\Omega_m^+} d\tilde{p}[\theta(yp)\, e^{-izp} a(p) + \theta(-yp)\, e^{izp} b^\dagger(p)].\end{aligned} \tag{5}$$

This extension actually turns out to be very natural. If z belongs to the *forward tube* $\mathscr{T}_+$ (i.e., y belongs to the open forward light cone V_+), then $yp > 0$ for all $p \in \Omega_m^+$, hence

$$\phi(z) = \int_{\Omega_m^+} d\tilde{p} e^{-izp} a(p) \tag{6}$$

is analytic and contains only the positive-frequency (annihilation) part of the field. If z belongs to the *backward tube* $\mathscr{T}_-$ (i.e., y belongs to the open backward light cone V_-), then $yp < 0$ for all $p \in \Omega_m^+$ and

$$\phi(z) = \int_{\Omega_m^+} d\tilde{p} e^{izp} b^\dagger(p) \tag{7}$$

is again analytic and contains only the negative-frequency (creation) part of the field. We define $\theta(0) = \frac{1}{2}$, so that for $y = 0$, Eq. (5) actually gives $\frac{1}{2}\phi(x)$. This is of no consequence since in computations only values of $\phi(z)$ in $\mathscr{T} \equiv \mathscr{T}_+ \cup \mathscr{T}_-$ will be used. Thus the field $\phi(z)$ defined by (5) exists (at least as an operator-valued distribution [8]) on *all* of $\mathbb{C}^4$ and is actually an analytic operator-valued *function* on $\mathcal{T}$. It has a discontinuity when crossing $\mathbb{R}^4$ in any timelike direction. Furthermore, a very suggestive integral representation of $\phi(z)$ in terms of $\phi(x)$ is obtained by noting that

$$\theta(u)e^{-u} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s+i} e^{-isu}. \tag{8}$$

Substituting (8) into (5) and exchanging the order of integration, we obtain

$$\phi(z) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s+i}\phi(x+sy). \tag{9}$$

This equation is dynamically useless since a complete knowledge of $\phi(x)$ is required to determine $\phi(z)$. But it displays the non-local dependence of the latter on the former in a precise and intuitive way and has the advantage that it avoids momentum space, thus opening the way to obvious generalizations. One immediate consequence is the illumination of the relation between *causality* and *analyticity*: if $y \in V_+$, the line $x(s) = x + sy$ is a possible world-line for a particle, whereas for $y \in V_-$ it is a possible world line for an antiparticle. This is related to the analyticity of $\phi(z)$, since only for $y \in V_\pm$ is the factor $\exp(\mp yp)$ in (5) a decaying exponential, providing an effective (but *natural*, hence covariant) momentum cutoff.

Note that Eq. (9) has a generality going far beyond the present context: it can be used (formally, at least) to extend any system of fields (classical or quantized, Bose or Fermi, free or interacting) to $\mathbb{C}^4$. Some of these ideas will be developed in Sections 3 and 4. For the time being, we continue with the free Klein–Gordon field.

Our first objective is to develop a phase-space interpretation of $\mathcal{T}$ and reconstruct the usual formalism in this framework. Thus, let $z = x - iy \in \mathcal{T}_+$. We claim that $\phi(z)^\dagger$ creates an extended particle which moves with expected energy-momentum proportional to y and is "focused" in spacetime about the event x. (It spreads in both timelike directions away from x as it must, being a solution of the Klein–Gordon equation.) For let $|0\rangle$ denote the vacuum state and define the one-particle state

$$e_z^+ = \phi(z)^\dagger\,|\,0\rangle = \int_{\Omega_m^+} d\tilde{p}\, e^{i\bar{z}p} a^\dagger(p)\,|\,0\rangle \equiv \int_{\Omega_m^+} d\tilde{p}\, e^{i\bar{z}p}\,|\,p^+\rangle. \tag{10}$$

Then for $q \in \Omega_m^+$,

$$\langle q^+ \,|\, e_z^+\rangle = e^{i\bar{z}q} = e^{ixq - yq} \tag{11}$$

which coincides, up to normalization, with e_z as defined in [3, Eq. (3.5)]. As shown there (Sect. 4), e_z is a (normalizable) state which gives rise to the following expectation values for the energy-momentum P_μ and the Newton–Wigner position operators $\mathbf{X}(x_0)$ at time x_0:

$$\langle P_\mu \rangle = A(\lambda)\, y_\mu \tag{12}$$

$$\langle \mathbf{X}(x_0) \rangle = \mathbf{x}, \tag{12'}$$

where $\lambda \equiv (y^2)^{1/2} > 0$ and

$$A(\lambda) = (m/\lambda)\, K_2(2\lambda m)/K_1(2\lambda m), \tag{13}$$

K_ν being the modified Bessel functions. The relation (12) between P_μ and y_μ can be understood intuitively as follows: for fixed $y \in V_+$, the factor e^{-yp} in (10) filters out all momenta $p \in \Omega_m^+$ which are widely different from $p_y \equiv (m/\lambda)\, y$, since $yp \geqslant \lambda m$, with equality only for $p = p_y$. This explains the factor (m/λ) in (13). The factor K_2/K_1 is of relativistic origin, i.e., due to the form of $\omega(\mathbf{p})$, and goes to unity as $\lambda m \equiv \lambda mc/\hbar \to \infty$. (In this limit, e^{-yp} becomes a Gaussian and the realization of states by holomorphic functions gives a representation of the Galilean group which is a close relative of the Klauder–Bargmann–Segal coherent-state representation of the Heisenberg group (see [3, Eq. (2.11) and Theorem3]). Thus the *direction* of y is seen to have a direct physical significance. As for its *magnitude* λ, note that the larger we take λ, the more peaked the exponential e^{-yp} becomes about its maximum at p_y, hence λ^{-1} roughly measures the resolution of e_z^+ in momentum space (see [3, Eqs. (4.6) and (4.8)] for a precise statement). Thus λ is a rough measure of the *spatial extension* of e_z^+ in its rest frame $(\mathbf{y} = 0)$ at the time x_0 when it is maximally focused at $\mathbf{x}$ (see [7, Fig. 1]).

Similarly, for $z \in \mathscr{T}_-$, form the one-antiparticle state

$$e_z^- = \phi(z) \mid 0 \rangle = \int_{\Omega_m^+} d\tilde{p}\, e^{izp} b^\dagger(p) \mid 0 \rangle \equiv \int_{\Omega_m^+} d\tilde{p}\, e^{izp} \mid p^- \rangle. \tag{14}$$

Since for $q \in \Omega_m^+$

$$\langle q^- \mid e_z^- \rangle = e^{izq} = \langle q^+ \mid e_{\bar{z}}^+ \rangle, \tag{15}$$

e_z^- has exactly the same spacetime behavior as $e_{\bar{z}}^+$. This corresponds to the usual notion that an antiparticle is a particle moving "backwards" in time. But whereas in $\mathbb{R}^4$ the idea of moving backwards is difficult to make precise without parametrizing the motion, it is quite natural in complex spacetime: choose $y \in V_-$ instead of $y \in V_+$. This is also evident from the more general formula (9).

Having established the significance of x and y for the states $e_z^\pm$, let us now in turn establish the significance of these states. We will use them to construct an appropriate generalization of the resolution of unity by coherent states [1–4]. Let S be a spacelike 3-surface in spacetime $\mathbb{R}^4$. Fix $\lambda > 0$, and let $\Omega_\lambda^\pm$ be the hyperboloids in $V_\pm$ with $y^2 = \lambda^2$. In view of Eqs. (12) and (15), the six-dimensional manifolds

$$\sigma_\pm = \{x - iy \in \mathscr{T}_\pm \mid x \in S,\ y \in \Omega_\lambda^\pm\} = S \times \Omega_\lambda^\pm \tag{16}$$

can be interpreted as *classical phase spaces* for single particles and antiparticles, respectively. Denote by Π_+ and Π_- the projection operators onto the one-particle and one-antiparticle subspaces of the Fock space. We will construct a positive, covariant measure $d\sigma$ on $\sigma_\pm$ such that $\Pi_\pm$ have the "coherent-state" representations

$$\Pi_\pm = \int_{\sigma_\pm} d\sigma \mid e_z^\pm \rangle\langle e_z^\pm \mid, \tag{17}$$

which shows that the states $e_z^\pm$ are a vital link between the classical and quantum particle pictures.

The canonical measure on any phase space is its *Liouville measure*, which is related to the *symplectic structure* of the space [10]. Since we insist on covariance, this suggests that we begin with the Poincaré-invariant 2-form

$$\alpha = dx_\mu \wedge dy^\mu \tag{18}$$

on $\mathbb{C}^4$. We wish to define $d\sigma$ as the positive measure obtained by restricting the 6-form

$$\alpha \wedge \alpha \wedge \alpha = 3!\, d\hat{x}^\mu \wedge d\hat{y}_\mu \tag{19}$$

to $\sigma_\pm$. The 3-forms $d\hat{x}^\mu$ and $d\hat{y}_\mu$ are the Hodge duals of dx_μ and dy^μ (see [4] for details), whose restrictions to $\sigma_\pm$ are

$$d\hat{x}^\mu = (n^\mu/n^0)\, d^3x \tag{20}$$

$$d\hat{y}_\mu = (y_\mu/y_0)\, d^3y. \tag{20'}$$

Here n^μ is the normal to S and we have identified $d\hat{x}^0$ and $d\hat{y}_0$ with the Lebesgue measures $d^3x \equiv dx_1\, dx_2\, dx_3$ and $d^3y = dy_1\, dy_2\, dy_3$ by choosing orientations ($\sigma_\pm$ are now parametrized by $\mathbf{x}$ and $\mathbf{y}$). Thus

$$d\hat{x}^\mu \wedge d\hat{y}_\mu = \left(1 - \frac{\mathbf{n}}{n_0} \cdot \frac{\mathbf{y}}{y_0}\right) d^3x\, d^3y, \tag{21}$$

which is indeed a positive measure on $\sigma_\pm$, since S is spacelike. We now define

$$d\sigma = C_\lambda\, d\hat{x}^\mu \wedge d\hat{y}_\mu \tag{22}$$

where

$$C_\lambda = m^2[\pi\lambda^2 K_2(2\lambda m)]^{-1} \tag{23}$$

is a normalization constant (this differs by a factor 2 from Eq. (3.12) in [3], due to the change from $d\Omega(\mathbf{p})$ to $d\tilde{p}$). It will often be convenient to think of (22) as a 6-form on $\mathbb{C}^4$ (in which case $d\sigma$ must not be confused with an exterior derivative!) rather than the measure on $\sigma_\pm$ obtained from it by restriction. The representations

(17) now follow from results derived in [2–4], namely that for p and q in Ω_m^+ and any σ_+ as above, we have

$$\int_{\sigma_+} d\sigma e^{i\bar{z}p - izq} = (2\pi)^3 \cdot 2\omega\, \delta(\mathbf{p}-\mathbf{q}) = \langle p \mid q \rangle \tag{24}$$

and its equivalent form obtained by replacing z by $\bar{z}$ and σ_+ by σ_-. In particular, this shows that the integral on the left is independent of σ. (For a rigorous proof of (24), see [3, Theorem 2; 4, Theorem 3].)

We have thus established the basic connection between complex spacetime and phase space for single particles. Clearly this can be extended to any number of particles and antiparticles, but we shall not pursue this line of thought here. Rather, note that y plays a similar role in the field $\phi(z)$ for $z \in \mathscr{T}$ (Eqs. (6) and (7)) as it did in $e_z^{\pm}$: namely, the factors $e^{\pm yp}$ tend to filter out all momenta widely different from p_y (for $y \in V_-$, define $p_y = -(m/\lambda)\, y$). It is important to understand that y itself should *not* be thought of as a four-momentum. For one thing, it has the physical dimensions of length and its magnitude λ is an arbitrary positive number. Rather, we think of y as a *control four-vector* for the energy-monentum, in precisely the same way as the inverse temperature β in statistical mechanics is a control parameter for the energy. (In fact, this analogy is very close and possibly very deep!)

We proceed now to build global observables such as the total charge and energy-momentum for the field $\phi(z)$, using Eq. (24) and its counterpart for σ_-. Thus the particle and antiparticle number operators are

$$N_+ = \int_{\Omega_m^+} d\tilde{p} a^\dagger(p)\, a(p) = \int_{\sigma_+} d\sigma \phi(z)^\dagger\, \phi(z) \tag{25}$$

$$N_- = \int_{\Omega_m^+} d\tilde{p} b^\dagger(p)\, b(p) = \int_{\sigma_-} d\sigma \phi(z)\, \phi(z)^\dagger. \tag{25'}$$

Although so far we have not needed to introduce normal (Wick) ordering, this can be done to give a compact expression for the charge operator,

$$Q = N_+ - N_- = \int_\sigma d\sigma : \phi(z)^\dagger\, \phi(z) :, \tag{26}$$

where the two-sheeted particle–antiparticle phase-space $\sigma \equiv \sigma_+ - \sigma_-$ is the union of σ_+ and σ_- but with the orientation of σ_- *reversed.* Note that Wick ordering in (26) can also be viewed as *reverse imaginary-time ordering,* if we define $\phi^\dagger(\bar{z}) = \phi(z)^\dagger$. It turns out that (26) can be cast into a form similar to the usual one [5] of the $\mathbb{R}^4$-theory,

$$Q_{\text{usual}} = i \int_S d\hat{x}^\mu : \phi(x)^\dagger\, \partial_\mu \phi(x) - \partial_\mu \phi(x)^\dagger \cdot \phi(x) :. \tag{27}$$

For this purpose, let

$$B_\lambda^\pm = \{ y \in V_\pm \mid y^2 > \lambda^2 \} \tag{28}$$

and $B_\lambda = B_\lambda^+ + B_\lambda^-$ (i.e., the union of B_λ^+ and B_λ^- with the *same* orientation). Since the outward normal of B_λ^+ points down and that of B_λ^- points up, the oriented boundary of B_λ is $\partial B_\lambda = -\Omega_\lambda^+ + \Omega_\lambda^- \equiv -\Omega_\lambda$. Thus (26) gives

$$Q = C_\lambda \int_S d\hat{x}^\mu \int_{-\partial B_\lambda} d\hat{y}_\mu : \phi^\dagger \phi := -C_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4y \frac{\partial}{\partial y^\mu} : \phi^\dagger \phi : \tag{29}$$

by Stokes' theorem. (The contributions to the integral from $y_0 \to \pm\infty$ vanish, as has been shown rigorously for the one-particle theory in [4, Theorem 3].) Now

$$-\frac{\partial}{\partial y^\mu} = i\left(\frac{\partial}{\partial z^\mu} - \frac{\partial}{\partial \bar{z}^\mu}\right) \equiv i(\partial_\mu - \bar{\partial}_\mu), \tag{30}$$

hence by the analyticity of $\phi(z)$,

$$\begin{aligned} Q &= iC_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4y \,{:}\phi^\dagger \partial_\mu \phi - \bar{\partial}_\mu \phi^\dagger \cdot \phi{:} \\ &\equiv C_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4y j_\mu(z) \equiv \int_S d\hat{x}^\mu J_\mu(x). \end{aligned} \tag{31}$$

This exhibits Q as the total flux through S of the spacetime current

$$J_\mu(x) = iC_\lambda \int_{B_\lambda} d^4y \,{:}\phi^\dagger \partial_\mu \phi - \bar{\partial}_\mu \phi^\dagger \cdot \phi{:}, \tag{32}$$

which is a *regularized version* of the usual current in (27). Note that $J_\mu(x)$ and $j_\mu(z)$ are both conserved since

$$\frac{\partial j_\mu}{\partial x_\mu} = -\frac{\partial^2}{\partial x_\mu \partial y^\mu} {:}\phi^\dagger \phi{:} = i(\partial^\mu + \bar{\partial}^\mu)(\partial_\mu - \bar{\partial}_\mu){:}\, \phi^\dagger \phi{:} \equiv i(\Box - \bar{\Box})\,{:}\phi^\dagger \phi{:} \tag{33}$$

vanishes by the analyticity of $\phi(z)$ combined with the Klein–Gordon equation.

The current $J_\mu(x)$ in (32) is thus non-locally related to the usual current (the latter is extended to $\mathscr{T}$ by extending ϕ, then integrated over B_λ). It may be preferable to consider $j_\mu(z)$ instead (Eq. (31)), which is "microlocal" in $\mathscr{T}$.

As can be easily verified directly,

$$[\phi(z'), Q] = \phi(z') \qquad \text{for all} \quad z' \in \mathbb{C}^4, \tag{34}$$

which states that $\phi(z')$ lowers the charge by a single unit. Substituting (26) for Q, we obtain

$$\phi(z') = \int_\sigma d\sigma [\phi(z'), {:}\phi(z)^\dagger \phi(z){:}] = \int_\sigma d\sigma [\phi(z'), \phi(z)^\dagger]\, \phi(z). \tag{35}$$

Thus for any $z' \in \mathbb{C}^4$,

$$\phi(z') = \int_\sigma d\sigma K(z', \bar{z})\, \phi(z), \tag{36}$$

where

$$K(z', \bar{z}) = [\phi(z'), \phi(z)^\dagger] = \int_{\Omega_m^+} d\tilde{p}[\theta(y'p)\, \theta(yp)\, e^{-i(z'-\bar{z})p} - \theta(-y'p)\, \theta(-yp)\, e^{i(z'-\bar{z})p}], \tag{37}$$

defined (as a distribution) on $\mathbb{C}^4 \times \mathbb{C}^4$, is piecewise analytic in $z' - \bar{z}$ when $z', z \in \mathscr{T}$, with

$$K(z', \bar{z}) = \begin{Bmatrix} -i\Delta^+(z'-\bar{z}),\ z', z \in \mathscr{T}_+ \\ i\Delta^-(z'-\bar{z}),\ z', z \in \mathscr{T}_- \\ 0,\ z' \in \mathscr{T}_+,\ z \in \mathscr{T}_- \\ 0,\ z' \in \mathscr{T}_-,\ z \in \mathscr{T}_+ \end{Bmatrix}. \tag{38}$$

The two-point functions $-i\Delta^+$ and $i\Delta^-$ are analytic in $\mathscr{T}_+$ and $\mathscr{T}_-$, respectively, and act as *reproducing kernels* for the one-particle and one-antiparticle spaces [3]. It is reasonable, by Eq. (36), to call $K(z', \bar{z})$ a reproducing kernel for the field $\phi(z)$, though this differs somewhat from the standard usage of the term as applied to Hilbert spaces.

Note the relations

$$-i\Delta^+(z'-\bar{z}) = \langle 0 \,|\, \phi(z')\, \phi(z)^\dagger \,|\, 0 \rangle = \langle e_{z'}^+ \,|\, e_z^+ \rangle, \qquad z', z \in \mathscr{T}_+ \tag{39}$$

$$i\Delta^-(z'-\bar{z}) = \langle 0 \,|\, \phi(z)^\dagger\, \phi(z') \,|\, 0 \rangle = \langle e_z^- \,|\, e_{z'}^- \rangle, \qquad z', z \in \mathscr{T}_-, \tag{39'}$$

giving the *analytic* (Wightman) two-point functions *directly* in terms of the vacuum expectation values (VEVs) of $\phi(z)$'s. In the usual theory [8], VEVs are first defined as distributions in terms of $\phi(x)$'s, then continued analytically using the spectral condition.

Equation (37) shows that K propagates positive frequencies into the forward tube (also called the future tube) and negative ones into the backward tube (past tube). This is somewhat reminiscent of the Feynman propagator, but K is a solution of the *homogeneous* Klein–Gordon equation rather than a Green function. The discontinuity occurs *across* $\mathbb{R}^4$ in the timelike y directions rather than *in* $\mathbb{R}^4$ along the timelike x directions. For this reason we prefer to call K a reproducing kernel rather than propagator.

The four-momentum and angular momentum can be expressed in a form similar to that for the charge,

$$P_\mu = i\int_\sigma d\sigma\, :\phi^\dagger \partial_\mu \phi: \tag{40}$$

$$M_{\mu\nu} = i\int_\sigma d\sigma\, :\phi^\dagger(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi:. \tag{40'}$$

Like Q, these expressions may be displayed as regularizations of the usual ones in $\mathbb{R}^4$.

3. The Free Dirac Field

Beginning with the free Dirac field [5] $\psi(x)$ over $\mathbb{R}^4$, we use (9) to extend it to $\mathbb{C}^4$:

$$\psi(z) = \int_{\Omega_m^+} d\tilde{p}[\theta(yp)\, e^{-izp} u^\alpha(p)\, b_\alpha(p) + \theta(-yp)\, e^{izp} v^\alpha(p)\, d_\alpha^\dagger(p)], \tag{41}$$

where u^α and v^α are positive- and negative-frequency spinors and a sum over the polarizations $\alpha = 1, 2$ is implied. The operators b_α and d_α satisfy the anticommutation relations

$$\{b_\alpha(p), b_\beta^\dagger(q)\} = \{d_\alpha(p), d_\beta^\dagger(q)\} = (2\pi)^3 \cdot 2\omega\, \delta(\mathbf{p} - \mathbf{q})\, \delta_{\alpha\beta} \tag{42}$$

on Ω_m^+, with all other anticommutators zero. The spinors satisfy the orthogonality and completeness relations

$$\bar{u}^\alpha(p)\, u^\beta(p) = -\bar{v}^\alpha(p)\, v^\beta(p) = \delta^{\alpha\beta} \tag{43}$$

$$u^\alpha(p) \otimes \bar{u}^\alpha(p) = (\not{p} + m)/2m \tag{44}$$

$$v^\alpha(p) \otimes \bar{v}^\alpha(p) = (\not{p} - m)/2m \tag{44'}$$

(sum over α in (44) and (44′)), as well as

$$\bar{u}^\alpha(p)\, \gamma_\mu u^\beta(p) = \bar{v}^\alpha(p)\, \gamma_\mu v^\beta(p) = (p_\mu/m)\, \delta^{\alpha\beta}. \tag{45}$$

To derive an expression for the charge Q in terms of $\psi(z)$, we begin with the momentum representation

$$Q = \int_{\Omega_m^+} d\tilde{p}[b_\alpha^\dagger(p)\, b_\alpha(p) - d_\alpha^\dagger(p)\, d_\alpha(p)] = N_+ - N_- . \tag{46}$$

Using (24) and (43), we find the compact expression

$$Q = \int_\sigma d\sigma :\overline{\psi(z)}\, \psi(z):, \tag{47}$$

where we emphasize that the integration over σ_- involves three changes of sign: one due to the orientation of σ_-, one due to Wick ordering, and one due to the second of Eqs. (43).

To relate (47) to the usual expression in $\mathbb{R}^4$,

$$Q_{\text{usual}} = \int_S d\hat{x}^\mu :\overline{\psi(x)}\, \gamma_\mu \psi(x):, \tag{48}$$

recall that $\Omega_\lambda = -\partial B_\lambda$, hence (47) gives

$$Q = C_\lambda \int_S d\hat{x}^\mu \int_{\Omega_\lambda} d\hat{y}_\mu :\bar{\psi}\psi: = -C_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4y \frac{\partial}{\partial y^\mu} :\bar{\psi}\psi:. \tag{49}$$

If we define

$$j_\mu(z) \equiv 2m :\overline{\psi(z)}\,\gamma_\mu \psi(z): \tag{50}$$

(the factor $2m$ gives j_μ the correct physical dimensions, given our normalization) then the Dirac equation, combined with the analyticity of ψ in $\mathscr{T}$, implies

$$\begin{aligned}\tfrac{1}{2} j_\mu(x) &= :\bar{\psi}\gamma_\mu i\gamma_\nu \partial^\nu \psi: = i\partial^\nu :\bar{\psi}\gamma_\mu\gamma_\nu\psi: \\ &= i\partial_\mu :\bar{\psi}\psi: + \partial^\nu :\bar{\psi}\sigma_{\mu\nu}\psi:,\end{aligned} \tag{51}$$

the real part of which gives the $\mathbb{C}^4$ version of the Gordon identity,

$$j_\mu(z) = i(\partial_\mu - \bar{\partial}_\mu) :\bar{\psi}\psi: + (\partial^\nu + \bar{\partial}^\nu) :\bar{\psi}\sigma_{\mu\nu}\psi:. \tag{52}$$

The two terms in (52) are conserved separately, and the second term, which is due to spin, does not contribute to the total charge since it is a pure divergence with respect to x. Thus (49), (52), and (30) give

$$Q = C_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4 y\, j_\mu(z) \equiv \int_S d\hat{x}^\mu J_\mu(x) \tag{53}$$

which is the desired expression with the regularized current

$$J_\mu(x) = C_\lambda \int_{B_\lambda} d^4 y\, j_\mu(x - iy). \tag{54}$$

Again, the relation

$$\psi(z') = [\psi(z'), Q], \qquad z' \in \mathbb{C}^4 \tag{55}$$

can be combined with (47) to give a reproducing kernel for ψ:

$$\psi(z') = \int_\sigma d\sigma \{\psi(z'), \overline{\psi(z)}\}\, \psi(z) \equiv \int_\sigma d\sigma K_D(z', \bar{z})\, \psi(z), \tag{56}$$

where K_D, defined as a distribution on $\mathbb{C}^4 \times \mathbb{C}^4$, is given by the matrix

$$\begin{aligned}K_D(z', \bar{z}) &= \{\psi(z'), \overline{\psi(z)}\} \\ &= \int_{\Omega_m^+} d\tilde{p} [\theta(y'p)\,\theta(yp)\, e^{-i(z'-\bar{z})p} u^\alpha \otimes \bar{u}^\alpha \\ &\quad + \theta(-y'p)\,\theta(-yp)\, e^{i(z'-\bar{z})p} v^\alpha \otimes \bar{v}^\alpha] \\ &= (2m)^{-1}(i\partial\!\!\!/' + m)\, K(z', \bar{z}),\end{aligned} \tag{57}$$

where the last step follows from Eqs. (44) and K is given by (37). Like K, K_D is piecewise analytic in $z' - \bar{z}$ for $z', z \in \mathscr{T}$. We also have from (53) and (55) the more complicated form

$$\psi(z') = 2mC_\lambda \int_S d\hat{x}^\mu \int_{B_\lambda} d^4y K_D(z', \bar{z}) \gamma_\mu \psi(z), \tag{58}$$

which is closer to the usual relation.

The momentum and angular momentum operators for the Dirac field are represented by

$$P_\mu = i \int_\sigma d\sigma :\bar{\psi}\partial_\mu \psi: \tag{59}$$

$$M_{\mu\nu} = \int_\sigma d\sigma :\bar{\psi}(ix_\mu \partial_\nu - ix_\nu \partial_\mu + \tfrac{1}{2}\sigma_{\mu\nu})\psi:. \tag{60}$$

More generally, let $\psi(z)$ represent either the Klein–Gordon field (in which case $\bar{\psi} \equiv \psi^\dagger$) or the Dirac field, and let T_a be the local generators of an arbitrary *interval or external* symmetry, so that the infinitesimal change in $\psi(z)$ is

$$\delta\psi(z) = -i\varepsilon^a T_a \psi(z). \tag{61}$$

Thus $T_a = 1$ for $U(1)$ gauge symmetry, $T_\mu = i\partial_\mu$ for spacetime translations (the derivative is with respect to x^μ but may also be taken with respect to z^μ by analyticity), etc. Then the corresponding global, conserved observable is the operator

$$Q_a = \int_\sigma d\sigma :\bar{\psi} T_a \psi:. \tag{62}$$

To show this, note that (62) implies

$$[\psi(z'), Q_a] = \int_a d\sigma K_D(z', \bar{z}) \, T_a \psi(z). \tag{63}$$

(If ψ is the Klein–Gordon field, replace K_D with K, Eq. (37).) Since T_a generates a symmetry, $T_a\psi(z)$ is a solution of the appropriate field equation and hence is reproduced by K_D, i.e.,

$$\int_\sigma d\sigma K_D(z', \bar{z}) \, T_a \psi(z) = T_a \psi(z'). \tag{64}$$

Hence Q_a has the required property

$$[\psi(z'), Q_a] = T_a \psi(z'). \tag{65}$$

It can furthermore be checked that

$$[Q_a, Q_b] = \int_\sigma d\sigma \, :\bar{\psi}[T_a, T_b]\psi:, \tag{66}$$

so that the mapping $T_a \mapsto Q_a$ preserves Lie brackets. Clearly (62) can be used to define the conserved charges of non-abelian (Yang–Mills) gauge theory when the field has the appropriate internal symmetry.

Finally, we show that due to the separation of positive and negative frequencies, the interference effect known as Zitterbewegung does not occur in our formalism. Let S_t be the configuration space defined by $x_0 = t$. Then according to Eq. (54), the components of the *total* three-current at time t are

$$J_k(t) = 2mC_\lambda \int_{S_t} d^3x \int_{B_\lambda} d^4y \, :\bar{\psi}\gamma_k\psi:. \tag{67}$$

Substituting (41) and using (45), we compute

$$J_k(t) = \int_{\Omega_m^+} d\tilde{p}(p_k/\omega)[b_\alpha^\dagger(p)\, b_\alpha(p) - d_\alpha^\dagger(p)\, d_\alpha(p)], \tag{68}$$

where we have used

$$C_\lambda \int_{B_\lambda^+} d^4y e^{-2yp} = 1. \tag{69}$$

Equation (68) shows that $J_k(t)$ is independent of time, hence Zitterbewegung does not occur. (The above computation can be repeated for the first-quantized Dirac wave function, giving a result identical to (68) except for a change of sign in the second term due to to the commutation of $d_\alpha^\dagger$ and d_α.)

4. Interacting Fields

Finally, we show that the relation between $\mathscr{T}$ and classical phase-space is still valid when interactions are present. We work within the axiomatic framework [8]. For simplicity we consider a single charged scalar field, though the arguments below easily generalize to an arbitrary system of mutually interacting fields. The field $\phi(x)$ is an operator-valued tempered distribution, hence possesses a (distributional) Fourier transform $\tilde{\phi}(p)$,

$$\phi(x) = \int_{\mathbb{R}^4} d^4p e^{-ixp}\tilde{\phi}(p). \tag{70}$$

In general, $\tilde{\phi}(p)$ does not vanish outside the light cone, so the integration is over $\mathbb{R}^4$. The relation between $p \in \mathbb{R}^4$ and the energy-momentum operators P_μ is obtained by considering the effect of infinitesimal translations,

$$i\partial_\mu \phi(x) = [\phi(x), P_\mu] \tag{71}$$

which in momentum space reads

$$[\tilde{\phi}(p), P_\mu] = p_\mu \tilde{\phi}(p). \tag{72}$$

This shows that $\tilde{\phi}(p)$ lowers the energy-momentum of a state by p and $\tilde{\phi}(p)^\dagger$ raises it by p. By assumption, the vacuum state is invariant under translations, hence

$$P_\mu \,|0\rangle = 0. \tag{73}$$

Thus

$$|p^+\rangle \equiv \tilde{\phi}(p)^\dagger |0\rangle \tag{74}$$

$$|p^-\rangle \equiv \tilde{\phi}(-p) |0\rangle \tag{74'}$$

are generalized (i.e., unnormalizable) states of precise energy-momentum p, if they do not vanish. Now the *spectral condition* requires that the joint spectrum Σ of the operators P_μ be contained in the closed forward light cone $\bar{V}_+$. Denote by Σ_1 the intersection of Σ with the support of $\tilde{\phi}(p)$. Then it follows that

$$|p^\pm\rangle = 0 \qquad \text{if} \quad p \notin \Sigma_1 \subset \bar{V}_+. \tag{75}$$

(For the free field of Sect. 2, $\Sigma_1 = \Omega_m^+$ and (75) reduces to $b(p)|0\rangle = a(p)|0\rangle = 0$ for all $p \in \Omega_m^+$.) Extending $\phi(x)$ to $\mathbb{C}^4$ by (9) gives

$$\phi(z) = \int_{\mathbb{R}^4} d^4p\, \theta(yp)\, e^{-izp} \tilde{\phi}(p). \tag{76}$$

For $z \in \mathscr{T}_+$ define the counterpart of (10),

$$e_z^+ = \phi(z)^\dagger |0\rangle = \int_{\mathbb{R}^4} d^4p\, \theta(yp)\, e^{i\bar{z}p} |p^+\rangle = \int_{\bar{V}_+} d^4p\, e^{i\bar{z}p} |p^+\rangle. \tag{77}$$

Although this is no longer a one-particle state (it does not have a definite mass, unless $\Sigma_1 = \Omega_m^+$, in which case the field is free), it shares many of the properties of its free counterpart. Since the P_μ are self-adjoint, we have

$$\langle p^+ \,|\, q^+\rangle = (2\pi)^{-3}\, \sigma(p^2)\, \delta(p-q), \tag{78}$$

where σ, a distribution with support in Σ_1, depends only on p^2 by Lorentz invariance. (If ϕ is the free field of mass m, then $\sigma(p^2)=\theta(p_0)\,\delta(p^2-m^2)=\delta(p^2-m^2)$ in $\bar{V}_+$.) Thus for $z', z \in \mathscr{T}_+$,

$$\begin{aligned}\langle e_{z'}^+ \mid e_z^+ \rangle &= \langle 0 \mid \phi(z')\,\phi(z)^\dagger \mid 0 \rangle \\ &= (2\pi)^{-3} \int_{\bar{V}_+} d^4p\,\sigma(p^2)\, e^{-i(z'-\bar{z})p} \\ &= \int_0^\infty dm^2 \sigma(m^2) \int_{\Omega_m^+} d\tilde{p}\, e^{-i(z'-\bar{z})p} \\ &= -i \int_0^\infty dm^2 \sigma(m^2)\, \Delta^+(z'-\bar{z}; m), \end{aligned} \tag{79}$$

where we have set $m^2 \equiv p^2$ and $\Delta^+(z'-\bar{z}; m)$ is given by (39). Equation (79) is the Källen–Lehmann representation [5] for the *analytically continued* two-point function of the field $\phi(x)$, which indicates that our extension $\phi(z)$ is closely related to the analytic continuation of VEVs in the axiomatic framework [8].

The expectation of P_μ in the state e_z^+ can now be computed. We assume that e_z^+ is normalizable, so that $\sigma(m^2)$ satisfies the regularity condition

$$\begin{aligned}\| e_z^+ \|^2 &= (2\pi)^{-3} \int_{\bar{V}_+} d^4p\,\sigma(p^2)\, e^{-2yp} \\ &= (4\pi^2\lambda)^{-1} \int_0^\infty dm \cdot m^2 \sigma(m^2)\, K_1(2\lambda m) \\ &\equiv F(\lambda) < \infty. \end{aligned} \tag{80}$$

It follows that

$$\begin{aligned}\langle e_z^+ \mid P_\mu e_z^+ \rangle &= (2\pi)^{-3} \int_{\bar{V}_+} d^4p\,\sigma(p^2)\, p_\mu e^{-2yp} = -\frac{1}{2}\frac{\partial F(\lambda)}{\partial y^\mu} \\ &= (-F'(\lambda)/2\lambda)\, y_\mu. \end{aligned} \tag{81}$$

Using the relation

$$-\frac{d}{d\xi}(\xi^{-1}K_n(\xi)) = \xi^{-1}K_{n+1}(\xi) \tag{82}$$

we find

$$\langle P_\mu \rangle = B(\lambda)\, y_\mu, \tag{83}$$

where

$$B(\lambda) = -F'(\lambda)/2\lambda F(\lambda) = \frac{\int_0^\infty dm \cdot m^2\sigma(m^2) \cdot (m/\lambda)\, K_2(2\lambda m)}{\int_0^\infty dm \cdot m^2 \sigma(m^2)\, K_1(2\lambda m)}. \tag{84}$$

Equations (83) and (84) generalize the relation (12), showing that y retains its interpretation as a control vector for the energy-momentum of the field even in the presence of interactions. (Clearly, the counterpart e_z^- of (14) can also be defined and has similar properties.) This suggest that the tube domains of axiomatic and constructive [11] quantum field theory also have physical interpretations in terms of phase space.

Incidentally, Eq. (84) shows that λ acts as an effective *ultraviolet cutoff*, since $K_n(2\lambda m)$ decays exponentially as $m \to \infty$.

5. Concluding Remarks

Although the free-field formalism developed here is largely equivalent to the usual one, it does differ from it in some fundamental ways:

(a) For Klein–Gordon particles, it gives rise to a *covariant and consistent* (i.e., nonnegative) *probability interpretation* of the theory [2, 3].

(b) In the Dirac case, the one-particle theory is also improved in that *no Zitterbewegung* occurs.

(c) Both of the above advantages result from the *added regularity* obtained by extending fields to $\mathbb{C}^4$, and especially to $\mathscr{T}$. Since products of fields are well defined in $\mathscr{T}$, there is good reason to hope that the new formalism can handle interactions better than the old.

Finally, we mention that there ought to be a connection between the restriction of $\phi(z)$ to the Euclidean region and the Euclidean field [11], though this connection is not yet clear.

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