# Generalized Wavelet Transforms. I.
# The Windowed X–Ray Transform

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ABSTRACT

The act of measuring a physical field (such as an electric or magnetic field, or the pressure distribution in space) suggests a certain integral transform which turns out to be a simple and natural generalization of the continuous wavelet transform to $n$ dimensions. A reconstruction formula is derived which inverts this transform. The $n$–dimensional objects correponding to wavelets are not tensor products of one–dimensional wavelets. Unlike such tensor products, they are natural with respect to the action of the affine group on $\mathbb{R}^n$. For $n = 1$, both the transform and the reconstruction formula reduce to their usual forms. The generalized transform is a windowed version of the X–ray transform (one–dimensional Radon transform in $\mathbb{R}^n$).

Key words: Wavelets , X–ray transform, Radon transform

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## 1. Definition of the Transform

The ideas presented here originated in relativistic quantum field theory (Kaiser [1987, 1990a]), where a method was developed to extend functions from $\mathbb{R}^n$ to $\mathbb{C}^n$ in a natural way. This gave rise to the "analytic–signal transform" (Kaiser [1990b]). Later it was realized that this is a special case of what we shall call a *windowed X–ray transform* and that the latter is an $n$–dimensional generalization of the (continuous) wavelet transform (Daubechies, Grossmann and Meyer [1986], Daubechies [1988], Meyer [1990]).

Let us suppose we wish to measure some physical field ("signal") distributed through $\mathbb{R}^n$, such as a pressure (sound) wave or an electromagnetic field. For simplicity, assume that the field is a *scalar,* i.e. real– valued, such as pressure. (Our considerations easily extend to vector– or tensor–valued fields, such as elecromagnetic fields.) Then the field is a function $f\colon \mathbb{R}^n \to \mathbb{R}$. We may think of $\mathbb{R}^n$ as physical space, in which case the field to be measured is time–independent, or as space–time, in which case it may be time–dependent. Actual measurements are never instantaneous, nor do they take place at just one point in space. A measurement must be performed by reading an instrument, and the instrument necessarily occupies some region in space and must interact with the field for some time–interval before giving a meaningful reading. In this paper, we assume that the instrument is sufficiently small that it can be assumed to be concentrated at a single point in space at any time. We allow our point–instrument to be in an arbitrary state of uniform motion, so that its position is given by $\mathbf{x}(t) = \mathbf{x} + t\mathbf{v}$, where $t \in \mathbb{R}$ is a "time" parameter and $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$. Note that $t$ need not be the physical time. For example, if $\mathbb{R}^n$ is space–time, then each "point" $\mathbf{x}$ corresponds to an *event,* i.e. a particular location in space at a particular time. In that case, the line $\mathbf{x}(t)$ is called a *world–line* and represents the entire history of the point–instrument traveling through space with uniform motion. The "velocity" vector $\mathbf{v}$ then has one too many components and may be regarded as a set of *homogeneous coordinates* for the physical velocity.*

Let us assume that the reading of the instrument at time $s$ gives weight $h(t-s)$

* The requirement that the latter cannot exceed the speed of light means that $\mathbf{v}$ must belong to the *light cone* in $\mathbb{R}^n$. However, these considerations do not concern us in this paper. They will become important in the sequel, where solutions of wave equations are analyzed.

to the value of the field at time $t$. Our model for the observed value of the field at the point $\mathbf{x}$, as measured by the instrument traveling with the uniform velocity $\mathbf{v}$, is then

$$f_h(\mathbf{x},\mathbf{v}) \equiv \int_{-\infty}^{\infty} dt\, h(t) f(\mathbf{x}+t\mathbf{v}).$$

In order to accomodate complex–valued signals $f(\mathbf{x})$, we now let the weight function $h(t)$ be complex–valued. (Similarly, vector–valued fields are analyzed with vector–valued weight functions, etc.) To minimize analytical subtleties, we assume that $h$ is a smooth, bounded function and $f$ is a smooth function with rapid decay (for example, a Schwartz test function).

**Definition.** The *windowed X–ray transform* (WXT) of $f\colon \mathbb{R}^n \to \mathbb{C}$ is the function $f_h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ given by

$$f_h(\mathbf{x},\mathbf{v}) = \int_{-\infty}^{\infty} dt\, h(t)^* f(\mathbf{x}+t\mathbf{v}), \tag{1.1}$$

where the asterisk denotes complex conjugation.

*Remarks.*

1. For $\mathbf{v}=\mathbf{0}$, the transform becomes trivial since $f_h(\mathbf{x},\mathbf{0}) = \hat{h}(0)^* f(\mathbf{x})$, where $\hat{h}$ is the Fourier transform of $h$. We will see that $\hat{h}(0)=0$ for "admissible" $h$, hence only $\mathbf{v}\in\mathbb{R}^n\backslash\{\mathbf{0}\}\equiv\mathbb{R}^n_*$ will be considered. This restriction is also necessary for physical reasons when $\mathbb{R}^n$ is space–time, since then $\mathbf{v}=\mathbf{0}$ would mean that the instrument does not experience the flow of time. Note also that $f_h$ has the following *dilation property* for $a\neq 0$:

$$f_h(\mathbf{x},a\mathbf{v}) = \int_{-\infty}^{\infty} dt\, |a|^{-1}\, h(t/a)^*\, f(\mathbf{x}+t\mathbf{v}) = f_{h_a}(\mathbf{x},\mathbf{v}) \tag{1.2}$$

   where $h_a(t)\equiv |a|^{-1}\, h(t/a)$.

2. For $n=1$ and $v\neq 0$, a change of variables gives

$$\begin{aligned} f_h(x,v) &= |v|^{-1}\int_{-\infty}^{\infty} dt'\, h\left(\frac{t'-x}{v}\right)^* f(t') \\ &= |v|^{-1/2}\, Wf(x,v), \end{aligned} \tag{1.3}$$

   where $Wf$ is the usual wavelet transform, with $v$ playing the role of a *dilation factor* and the window function $h(t)$ playing the role of a *basic wavelet.*

3 In the special case $h(t) \equiv 1$ and $|\mathbf{v}| = 1$, $f_h$ is known as the *X–ray transform* of $f$ (Helgason [1984]); it may then be regarded as being defined on the set of all affine lines in $\mathbb{R}^n$.

It will be useful to write $f_h$ in another form by substituting the Fourier representation of $f$ into $f_h$. Formally, this gives

$$\begin{aligned} f_h(\mathbf{x}, \mathbf{v}) &= \int_{-\infty}^{\infty} dt \int d^n\mathbf{p}\, e^{2\pi i \mathbf{p}\cdot(\mathbf{x}+t\mathbf{v})}\, h(t)^*\, \hat{f}(\mathbf{p}) \\ &= \int d^n\mathbf{p}\, e^{2\pi i \mathbf{p}\cdot\mathbf{x}}\, \hat{h}(\mathbf{p}\cdot\mathbf{v})^*\, \hat{f}(\mathbf{p}) \\ &\equiv \langle\, \hat{h}_{\mathbf{x},\mathbf{v}}\,, \hat{f}\,\rangle_{L^2} = \langle\, h_{\mathbf{x},\mathbf{v}}\,, f\,\rangle_{L^2}, \end{aligned} \tag{1.4}$$

where $\hat{h}_{\mathbf{x},\mathbf{v}}$ is defined by

$$\hat{h}_{\mathbf{x},\mathbf{v}}(\mathbf{p}) = e^{-2\pi i \mathbf{p}\cdot\mathbf{x}}\, \hat{h}(\mathbf{p}\cdot\mathbf{v}), \tag{1.5}$$

so that

$$h_{\mathbf{x},\mathbf{v}}(\mathbf{x}') = \int d^n\mathbf{p}\, e^{2\pi i \mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}\, \hat{h}(\mathbf{p}\cdot\mathbf{v}). \tag{1.6}$$

(We have adopted the convention used in the physics literature, where complex inner products are linear in the *second* factor and antilinear in the first factor.)

The functions $h_{\mathbf{x},\mathbf{v}}$ are our $n$–dimensional wavelets and will be used in the next section to reconstruct the signal $f$. Note that $\hat{h}_{\mathbf{x},\mathbf{v}}$ (hence also $h_{\mathbf{x},\mathbf{v}}$) is not square–integrable for $n > 1$, since its modulus is constant along directions orthogonal to $\mathbf{v}$. But eq. (1.4) still makes sense provided $\hat{f}$ is sufficiently well–behaved. (This is one of the reasons we have assumed that $f$ is a Schwartz test function.)

The usual way in which wavelets are generalized to higher dimensions is by taking tensor products of one–dimensional wavelets (Daubechies [1988]). However, this means that not all directions in $\mathbb{R}^n$ are treated equally, and consequently the set of generalized wavelets does not transform "naturally" (in a sense to be explained below) under the *affine group* $G$ of $\mathbb{R}^n$, which consists of all transformations of the form

$$\mathbf{x} \mapsto g(A, \mathbf{b})\mathbf{x} \equiv A\mathbf{x} + \mathbf{b} \tag{1.7}$$

with $A$ a non–singular $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$. Each such $g(A, \mathbf{b})$ defines a unitary operator on $L^2(\mathbb{R}^n)$, given by

$$(U(A,\mathbf{b})f)(\mathbf{x}) \equiv |A|^{-\frac{1}{2}} f\left(A^{-1}(\mathbf{x}-\mathbf{b})\right), \tag{1.8}$$

where $|A|$ denotes the absolute value of the determinant of $A$. The map $g(A, \mathbf{b}) \mapsto U(A, \mathbf{b})$ forms a *representation* of $G$ on $L^2(\mathbb{R}^n)$, meaning that it preserves the group structure of $G$ under compositions. To see how $h_{\mathbf{x},\mathbf{v}}$ transforms under $U$, note that the unitarity of $U$ implies

$$\begin{aligned}\langle\, U(A,\mathbf{b})\, h_{\mathbf{x},\mathbf{v}}\,, f\,\rangle &= \langle\, h_{\mathbf{x},\mathbf{v}}\,, U(A,\mathbf{b})^{-1} f\,\rangle \\ &= \int_{-\infty}^{\infty} dt\; h(t)^*\, |A|^{\frac{1}{2}} f\left(A(\mathbf{x}+t\mathbf{v})+\mathbf{b}\right) \\ &= |A|^{\frac{1}{2}} \langle\, h_{A\mathbf{x}+\mathbf{b},A\mathbf{v}}\,, f\,\rangle.\end{aligned} \tag{1.9}$$

Hence

$$U(A,\mathbf{b})\, h_{\mathbf{x},\mathbf{v}} = |A|^{\frac{1}{2}} h_{A\mathbf{x}+\mathbf{b},A\mathbf{v}}, \tag{1.10}$$

which states that affine transformations take wavelets to wavelets. Thus, for example, translations, rotaions and dilations merely translate, rotate and dilate the labels $\{\mathbf{x}, \mathbf{v}\}$, while the factor $|A|^{\frac{1}{2}}$ preserves unitarity. By contrast, tensor products of one–dimensional wavelets are not transformed into one another by rotations.

## 2. A Reconstruction Formula

A reconstruction consists of a recovery of $f$ from $f_h$ or its restriction to some subset. In the one–dimensional case, for example, $f$ can be reconstructed using *all* of $\mathbb{R} \times \mathbb{R}_*$ or (for certain choices of $h$) just a discrete subset (Daubechies [1988]). For general $n$, the choice of reconstructions becomes even richer since various new possibilities arise. For example, $h$ may have symmetries which imply that $f_h$ is determined by its values on some lower–dimensional subsets of $\mathbb{R}^n \times \mathbb{R}^n_*$, making integration over the whole space both unnecessary and undesirable. Furthermore, $f(\mathbf{x})$ may satisfy some partial differential equation which implies that $f$ is determined by its values on subsets of $\mathbb{R}^n$. For example, a pressure wave or the components of an electromagnetic field satisfy the wave equation away from sources, so that $f$ is determined by initial data on a Cauchy surface in $\mathbb{R}^n$ and it becomes, once more, unnecessary and undesirable to use all of $\mathbb{R}^n \times \mathbb{R}^n_*$ in the reconstruction.

The reconstruction to be developed in this section is "generic" in that it does not assume any particular forms for $h(t)$ or $f(\mathbf{x})$. It uses all of $\mathbb{R}^n \times \mathbb{R}^n_*$, consequently it breaks down for certain choices of $h$ or $f$. Again we emphasize that this is far from the only way to proceed; other types of reconstruction will be discussed in the sequel. The present reconstruction formula is interesting in part because it generalizes the one for the ordinary continuous wavelet transform ($n = 1$).

To reconstruct $f$, we look for a *resolution of unity* in terms of the vectors $h_{\mathbf{x},\mathbf{v}}$. This means we need a measure $d\mu(\mathbf{x}, \mathbf{v})$ on $\mathbb{R}^n \times \mathbb{R}^n_*$ such that

$$\int d\mu(\mathbf{x}, \mathbf{v})\, |f_h(\mathbf{x}, \mathbf{v})|^2 = \int d^n\mathbf{x}\, |f(\mathbf{x})|^2 \equiv \|f\|^2_{L^2}. \tag{2.1}$$

For then the map $T: f \mapsto f_h$ is an isometry from $L^2(\mathbb{R}^n)$ onto its range in $L^2(d\mu)$, and polarization gives

$$\langle\, g, T^*Tf\,\rangle \equiv \langle\, Tg, Tf\,\rangle = \langle\, g, f\,\rangle. \tag{2.2}$$

This shows that $f = T^*Tf = T^*f_h$ in $L^2(\mathbb{R}^n)$, which is the desired reconstruction formula. (See Kaiser [1990a] for background on resolutions of unity, generalized frames and related subjects.)

To obtain a resolution of unity, note that

$$f_h(\mathbf{x}, \mathbf{v}) = \Big(\hat{h}(\mathbf{p}\cdot\mathbf{v})^*\, \hat{f}(\mathbf{p})\Big)\check{\ }(\mathbf{x}), \tag{2.3}$$

where $\check{\ }$ denotes the inverse Fourier transform, so by Plancherel's theorem,

$$\int d^n\mathbf{x}\, |f_h(\mathbf{x}, \mathbf{v})|^2 = \int d^n\mathbf{p}\, |\hat{h}(\mathbf{p}\cdot\mathbf{v})|^2\, |\hat{f}(\mathbf{p})|^2. \tag{2.4}$$

We therefore need a measure $d\rho(\mathbf{v})$ on $\mathbb{R}^n_*$ such that

$$H(\mathbf{p}) \equiv \int d\rho(\mathbf{v})\, |\hat{h}(\mathbf{p}\cdot\mathbf{v})|^2 \equiv 1 \quad \text{for almost all } \mathbf{p}, \tag{2.5}$$

since then $d\mu(\mathbf{x}, \mathbf{v}) = d^n\mathbf{x}\, d\rho(\mathbf{v})$ has the desired property. The solution is simple: Every $\mathbf{p} \neq \mathbf{0}$ can be transformed to $\mathbf{q} \equiv (1, 0, \cdots 0)$ by a *dilation and rotation* of $\mathbb{R}^n$. That is, the orbit of $\mathbf{q}$ (in Fourier space) under dilations and rotations is $\mathbb{R}^n_*$. Thus we choose $d\rho$ to be invariant under rotations and dilations, which gives

$$d\rho(\mathbf{v}) = N|\mathbf{v}|^{-n} d^n\mathbf{v}, \tag{2.6}$$

where $N$ is a normalization constant and $|\mathbf{v}|$ is the Euclidean norm of $\mathbf{v}$. Then for $\mathbf{p} \neq \mathbf{0}$,

$$\begin{aligned} H(\mathbf{p}) = H(\mathbf{q}) &= N \int |\mathbf{v}|^{-n} d^n\mathbf{v}\, |\hat{h}(v_1)|^2 \\ &= N \int_{-\infty}^{\infty} dv_1\, |\hat{h}(v_1)|^2 \int_{\mathbb{R}^{n-1}} \frac{dv_2 \cdots dv_n}{(v_1^2 + \cdots v_n^2)^{n/2}}. \end{aligned} \tag{2.7}$$

But a straightforward computation gives

$$\int_{\mathbb{R}^{n-1}} \frac{dv_2 \cdots dv_n}{(v_1^2 + \cdots v_n^2)^{n/2}} = \frac{\pi^{n/2}}{|v_1|\, \Gamma(n/2)}. \tag{2.8}$$

This shows that the measure $d\mu(\mathbf{x}, \mathbf{v}) \equiv d^n\mathbf{x}\, d\rho(\mathbf{v})$ gives a resolution of unity if and only if

$$c_h \equiv \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\hat{h}(\xi)|^2 < \infty, \tag{2.9}$$

which is precisely the *admissibility condition* for the usual (one–dimensional) wavelet transform. If $h$ is admissible, the normalization constant is given by

$$N = \frac{\Gamma(n/2)}{\pi^{n/2}\, c_h}. \tag{2.10}$$

The reconstruction formula is

$$f(\mathbf{x}') = (T^* f_h)(\mathbf{x}') = N \int \frac{d^n\mathbf{x}\, d^n\mathbf{v}}{|\mathbf{v}|^n}\, h_{\mathbf{x},\mathbf{v}}(\mathbf{x}')\; f_h(\mathbf{x}, \mathbf{v}). \tag{2.11}$$

The *sense* in which this formula holds depends on the behavior of $f$. The class of possible $f$'s, in turn, depends on the choice of $h$. Note that in spite of the factor $|\mathbf{v}|^n$ in the denominator, there is no problem at $\mathbf{v} = \mathbf{0}$ since $f_h(\mathbf{x}, \mathbf{0}) = \hat{h}(0)^* f(\mathbf{x}) = 0$ by the admissibility condition, and a similar analysis can be made for small $|\mathbf{v}|$ by using the dilation property (eq. (1.2)).

*Remarks:*

1. We have assumed that the instrument has no spatial extension. A general instrument can be represented by a window $g : \mathbb{R}^n \to \mathbb{C}$, and its "motions" under

the affine group are given by applying the unitary operators $U(A,\mathbf{b})$ of eq. (1.8). Hence, the transform corresponding to $f_h$ becomes

$$\begin{aligned} f_g(\mathbf{x},A) &= \langle\, U(A,\mathbf{x})g, f\,\rangle = \langle\, g, U(A,\mathbf{x})^{-1} f\,\rangle \\ &= |A|^{\frac{1}{2}} \int_{\mathbb{R}^n} d^n\mathbf{t}\, g(\mathbf{t})^* \, f(\mathbf{x}+A\mathbf{t}). \end{aligned} \tag{2.12}$$

In the special case when $g$ is a point–instrument, its motion in space is a line which can be identified with the $x_1$–axis. We therefore take

$$g(\mathbf{t}) = h(t_1)\,\delta(t_2)\,\delta(t_3)\cdots\delta(t_n), \tag{2.13}$$

and the above reduces to

$$f_g(\mathbf{x},A) = |A|^{\frac{1}{2}} \int_{-\infty}^{\infty} dt\, h(t)^*\, f(\mathbf{x}+A\mathbf{t}) = |A|^{\frac{1}{2}} f_h(\mathbf{x},\mathbf{v}), \tag{2.14}$$

where we have set $\mathbf{t} \equiv (t,0,\cdots 0)$ and $A\mathbf{t} \equiv \mathbf{v}t$ (i.e., $\mathbf{v}$ is the first column of $A$), and $f_h$ is the WXT of $f$. $d$–dimensional instruments ($0 \le d \le n-1$) similarly define windowed versions of the *d–dimensional Radon transform* in $\mathbb{R}^n$ (Helgason [1984]).

2. I have recently learned of some interesting work by Holschneider [1990] along similar lines. He considers a two–dimensional wavelet transform which is covariant under the affine group of $\mathbb{R}^2$, corresponding to $n = 2$ above. When $g$ is supported on a line, say $g(t_1,t_2) = \delta(t_2)$, this becomes the one–dimensional Radon transform in $\mathbb{R}^2$. He then inverts this transform using a *generalized* wavelet reconstruction formula, where one basic wavelet $g$ is used in the decomposition and another $h$ is used in the reconstruction. The admissibility condition then becomes a joint condition for $h$ and $g$ corresponding to a "polarization" of eq. (2.9), and the singular behavior of $g$ can be compensated with regular behavior of $h$.

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