Generalized Wavelet Transforms. II.
The Multivariate Analytic–Signal Transform

Gerald Kaiser
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Mathematics Department
University of Lowell
Lowell, MA 01854
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ABSTRACT

The idea of an analytic signal associated with a function of one variable is generalized to an arbitrary number of variables. The proposed "analytic–signal transform" is an example of the windowed X–ray transform introduced earlier, which in turn is a multidimensional generalization of the wavelet transform. The analytic–signal transform, which is defined through a Cauchy–like integral, extends an arbitrary function $f(\mathbf{x})$ on $\mathbb{R}^n$ to a function $\tilde{f}(\mathbf{x}+i\mathbf{y})$ on $\mathbb{C}^n$ in a "partially analytic" way. $\tilde{f}(\mathbf{x}+i\mathbf{y})$ can be expressed as a (multivariate) Fourier–Laplace transform of the Fourier transform $\hat{f}$ of f. It has a jump across $\mathbb{R}^n$ equal to the (multivariate) Hilbert transform $H_{\mathbf{y}}f$ of f in the direction $\mathbf{y}$. When $\hat{f}$ has support in a convex cone V_+, $\tilde{f}(\mathbf{x}+i\mathbf{y})$ is analytic in the *tube* $\mathcal{T}_+ = \{\mathbf{x}+i\mathbf{y} \,|\, \mathbf{y} \in V_+'\}$, where V_+' is the cone *dual* to V_+. A similar situation occurs when $\hat{f}$ has support in a double cone $V = V_+ \cup \{-V_+\}$, which is the case, for example, when f is a solution of a wave equation. Then $\tilde{f}(\mathbf{x}+i\mathbf{y})$ is analytic in a double tube $\mathcal{T} = \mathcal{T}_+ \cup \mathcal{T}_-$. This implies, for instance, that the analytic–signal transform $\tilde{f}(\mathbf{x}+i\mathbf{y})$ of an electromagnetic wave $f(\mathbf{x})$ (defined over space–time) is analytic in $\mathbf{x}+i\mathbf{y}$ whenever $\mathbf{y}$ is in the forward– or backward light cone. Solutions of the Klein–Gordon equation have a natural decomposition in terms of "wavelets" $e_{\mathbf{x}+i\mathbf{y}}$ whose parameters $\mathbf{x}$ and $\mathbf{y}$ have a direct geometric interpretation. It is conjectured that a similar decomposition exists for solutions of the wave equation, which may be useful in the analysis of signals such as sound waves and electromagnetic fields.

Key words: analytic signal, Hilbert transform, Fourier–Laplace transform, Hardy spaces, wave equation, wavelets.
AMS(MOS) subject classification: 32, 35, 44

1. Analytic Signals in One Dimension

Suppose we are given a one–dimensional "signal," i.e. a real– or complex–valued function f of a single real variable x ("time"). To begin with, assume that f is smooth with rapid decay (e.g., a Schwartz test function), although many of our considerations will extend to certain kinds of distributions. Consider the positive– and negative– frequency parts of f, defined by

$$\begin{aligned} f_+(x) &\equiv \int_0^\infty dp\, e^{2\pi ipx}\, \hat{f}(p) \\ f_-(x) &\equiv \int_{-\infty}^0 dp\, e^{2\pi ipx}\, \hat{f}(p), \end{aligned} \tag{1.1}$$

where $\hat{}$ denotes the Fourier transform. Then f_+ and f_- extend analytically to the upper–half and lower–half complex planes, respectively, i.e.

$$\begin{aligned} f_+(x+iy) &= \int_0^\infty dp\, e^{2\pi ip(x+iy)}\, \hat{f}(p), \quad y>0 \\ f_-(x+iy) &= \int_{-\infty}^0 dp\, e^{2\pi ip(x+iy)}\, \hat{f}(p), \quad y<0, \end{aligned} \tag{1.2}$$

since the factor $e^{-2\pi py}$ decays rapidly for $p \to \pm\infty$ in the respective integrals. f_+ and f_- are just the (inverse) *Fourier–Laplace transforms* of the restrictions of $\hat{f}$ to the positive and negative frequencies. If f is complex–valued, then f_+ and f_- are independent and the original signal can be recovered from them as

$$f(x) = \lim_{y\downarrow 0} \left[f_+(x+iy) + f_-(x-iy) \right]. \tag{1.3}$$

If f is real–valued, then

$$\hat{f}(p) = \hat{f}(-p)^*, \tag{1.4}$$

where the asterisk denotes complex conjugation; hence f_+ and f_- are related by reflection,

$$f_+(x+iy) = f_-(x-iy)^*, \quad y>0, \tag{1.5}$$

and

$$f(x) = 2\lim_{y\downarrow 0} \Re f_+(x+iy) = 2\lim_{y\downarrow 0} \Re f_-(x-iy). \tag{1.6}$$

When f is real, the function $f_+(z)$ is known as the *analytic signal* associated with $f(x)$ (Gabor [1946]). A complex–valued signal would have *two* independent associated analytic signals f_+ and f_-. What significance do $f_\pm$ have? For one thing, they are *regularizations* of f. Eq. (1.3) states that f is jointly a "boundary–value" of the pair f_+ and f_-. As such, f may actually be quite singular while remaining the boundary–value of analytic functions. Also, $2f_\pm$ provide a kind of "envelope" description of f (see Born and Wolf [1975], Klauder and Sudarshan [1968]). For example, if $f(x) = \cos x$, then $2f_\pm(z) = e^{\pm iz}$.

In order to extend the concept of analytic signals to more than one dimension, let us first of all unify the definitions of f_+ and f_- by defining

$$\tilde{f}(x+iy) \equiv \int_{-\infty}^{\infty} dp\, \theta(py)\, e^{2\pi ip(x+iy)}\, \hat{f}(p) \tag{1.7}$$

for *arbitrary* $x+iy \in \mathbb{C}$, where θ is the unit step function, defined by

$$\theta(u) = \begin{cases} 0, & u < 0 \\ 1/2, & u = 0 \\ 1, & u > 0. \end{cases} \tag{1.8}$$

Then we have

$$\tilde{f}(x+iy) = \begin{cases} f_+(x+iy), & y > 0 \\ \frac{1}{2} f(x), & y = 0 \\ f_-(x+iy), & y < 0. \end{cases} \tag{1.9}$$

Although this unification of f_+ and f_- may at first appear to be somewhat artificial, it turns out to be quite natural, as will now be seen. Note first of all that for any real u, we have

$$\theta(u)\, e^{-2\pi u} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i}\, e^{2\pi i\tau u}, \tag{1.10}$$

since the contour on the right–hand side may be closed in the upper–half plane when $u > 0$ and in the lower–half plane when $u < 0$. For $u = 0$, the equation states that

$$\begin{aligned} \theta(0) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(\tau + i)\, d\tau}{\tau^2 + 1} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\tau^2 + 1} = \frac{1}{2}, \end{aligned} \tag{1.11}$$

in agreement with our definition, if we interpret the integral as the limit as $L \to \infty$ of the integral from $-L$ to L. Therefore

$$\theta(py)\, e^{2\pi i p(x+iy)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i}\, e^{2\pi i p(x+\tau y)}. \tag{1.12}$$

If this is substituted into our expression for $\tilde{f}(z)$ and the order of integrations on τ and p is exchanged, we obtain

$$\tilde{f}(x+iy) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i}\, f(x+\tau y) \tag{1.13}$$

for arbitrary $x + iy \in \mathbb{C}$. We shall refer to the right–hand side as the *analytic–signal transform* of $f(x)$. It bears a close relation to the *Hilbert transform*, which is defined by

$$Hf(x) = \frac{1}{\pi} \mathrm{PV} \int_{-\infty}^{\infty} \frac{du}{u}\, f(x-u), \tag{1.14}$$

where PV denotes the principal value of the integral. Consider the complex combination

$$\begin{aligned} f(x) - iHf(x) &= \frac{1}{\pi i} \int_{-\infty}^{\infty} du \left[\pi i \delta(u) + \mathrm{PV}\frac{1}{u} \right] f(x-u) \\ &= \frac{1}{\pi i} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{du}{u - i\epsilon}\, f(x-u) \\ &= \frac{1}{\pi i} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \frac{d\tau}{\tau - i}\, f(x - \tau\epsilon) \\ &= 2 \lim_{\epsilon \downarrow 0} \tilde{f}(x - i\epsilon). \end{aligned} \tag{1.15}$$

Similarly,

$$f(x) + iHf(x) = 2 \lim_{\epsilon \downarrow 0} \tilde{f}(x + i\epsilon). \tag{1.16}$$

Hence

$$Hf(x) = \frac{1}{i} \lim_{\epsilon \downarrow 0} [\tilde{f}(x+i\epsilon) - \tilde{f}(x-i\epsilon)], \tag{1.17}$$

which for real–valued f reduces to

$$Hf(x) = 2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(x+i\epsilon) = -2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(x-i\epsilon). \tag{1.18}$$

2. Generalization to n Dimensions

We are now ready to generalize the idea of analytic signals to an arbitrary number of dimensions. To minimize analytical complications, we assume initially that $f(\mathbf{x})$ belongs to the space of Schwartz test functions $\mathcal{S}(\mathbb{R}^n)$, although this assumption proves to be unnecessary.

Definition. The *analytic–signal transform* (AST) of $f(\mathbf{x})$ is defined by

$$\tilde{f}(\mathbf{x}+i\mathbf{y}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\tau}{\tau - i}\, f(\mathbf{x}+\tau\mathbf{y}). \tag{2.1}$$

The same argument as above shows that for $\mathbf{z}=\mathbf{x}+i\mathbf{y}\in\mathbb{C}^n$,

$$\begin{aligned}\tilde{f}(\mathbf{z}) &= \int_{\mathbb{R}^n} d^n\mathbf{p}\;\theta(\mathbf{p}\cdot\mathbf{y})\, e^{2\pi i\mathbf{p}\cdot\mathbf{z}}\,\hat{f}(\mathbf{p})\\ &= \int_{M_{\mathbf{y}}} d^n\mathbf{p}\, e^{2\pi i\mathbf{p}\cdot\mathbf{z}}\,\hat{f}(\mathbf{p}),\end{aligned} \tag{2.2}$$

where $M_{\mathbf{y}}$ is the half–space

$$M_{\mathbf{y}} \equiv \{\mathbf{p}\in\mathbb{R}^n \mid \mathbf{p}\cdot\mathbf{y}\ge 0\}, \qquad \mathbf{y}\neq\mathbf{0}. \tag{2.3}$$

We shall refer to the right–hand side of eq. (2.2) as the (inverse) *Fourier–Laplace transform* of $\hat{f}$ in $M_{\mathbf{y}}$. The integral converges absolutely whenever $\hat{f}\in L^1(\mathbb{R}^n)$, since $|e^{2\pi i\mathbf{p}\cdot\mathbf{z}}|\le 1$ on $M_{\mathbf{y}}$, defining $\tilde{f}$ as a *function* on $\mathbb{C}^n$, although not an analytic one in general (see below). This shows that $\tilde{f}(\mathbf{z})$ can actually be defined for some distributions f, not only for test functions.

Note: In spite of the appearance of expressions such as $\mathbf{p}\cdot\mathbf{y}$, we have not assumed any particular metric structure on $\mathbb{R}^n$. The Fourier transform naturally takes functions on $\mathbb{R}^n$ to functions on the *dual* space $\mathbb{R}_n \equiv (\mathbb{R}^n)^*$ of linear functionals, and $\mathbf{p}\cdot\mathbf{y}$ merely denotes the value $\mathbf{p}(\mathbf{y})$. This remark becomes important when considering time–dependent signals, so that $\mathbb{R}^n$ is space–time, for then the natural structure on $\mathbb{R}^n$ is a Lorentzian metric rather than a Euclidean metric.

For $n=1$, $\tilde{f}(z)$ was analytic in the upper– and lower–half planes. In more than one dimension, $\tilde{f}(\mathbf{z})$ need not be analytic, even though, for brevity, we still write it as a function of $\mathbf{z}$ rather than $\mathbf{z}$ and its complex conjugate $\mathbf{z}^*$. However, $\tilde{f}(\mathbf{z})$ does in general possess a *partial* analyticity which reduces to the above when $n = 1$. Consider the partial derivative of $\tilde{f}(\mathbf{z})$ with respect to $z_k^* \equiv x_k - iy_k$, defined by

$$2\bar{\partial}_k \tilde{f} \equiv 2\frac{\partial \tilde{f}}{\partial z_k^*} \equiv \frac{\partial \tilde{f}}{\partial x_k} + i\frac{\partial \tilde{f}}{\partial y_k}. \tag{2.4}$$

Then $\tilde{f}$ is analytic at $\mathbf{z}$ if and only if $\bar{\partial}_k \tilde{f} = 0$ for all k. But using our definition of $\tilde{f}(\mathbf{z})$, we find that

$$2\bar{\partial}_k \tilde{f}(\mathbf{z}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, \frac{\partial f}{\partial x_k}(\mathbf{x} + \tau\mathbf{y}). \tag{2.5}$$

It follows that the complex $\bar{\partial}$–derivative in the direction of y vanishes, i.e.

$$\begin{aligned} 4\pi \sum_k y_k\, \bar{\partial}_k \tilde{f}(\mathbf{z}) &= \int_{-\infty}^{\infty} d\tau\, \sum_k y_k \frac{\partial f}{\partial x_k}(\mathbf{x} + \tau\mathbf{y}) \\ &= \int_{-\infty}^{\infty} d\tau\, \frac{\partial}{\partial \tau} f(\mathbf{x} + \tau\mathbf{y}) = 0, \end{aligned} \tag{2.6}$$

if f decays for large $|\mathbf{x}|$ (e.g., if f is a test function, as we have assumed). Equivalently, using

$$\begin{aligned} 2\bar{\partial}_k\, \left[\theta(\mathbf{p}\cdot\mathbf{y})\, e^{2\pi i \mathbf{p}\cdot\mathbf{z}}\right] &= 2\bar{\partial}_k\, \left[\theta(\mathbf{p}\cdot\mathbf{y})\right]\, e^{2\pi i \mathbf{p}\cdot\mathbf{z}} \\ &= i\frac{\partial\theta(\mathbf{p}\cdot\mathbf{y})}{\partial y_k}\, e^{2\pi i \mathbf{p}\cdot\mathbf{z}} \\ &= ip_k\, \delta(\mathbf{p}\cdot\mathbf{y})\, e^{2\pi i \mathbf{p}\cdot\mathbf{z}} \\ &= ip_k\, \delta(\mathbf{p}\cdot\mathbf{y})\, e^{2\pi i \mathbf{p}\cdot\mathbf{x}}, \end{aligned} \tag{2.7}$$

we have for $\mathbf{y} \neq \mathbf{0}$

$$2\sum_k y_k\, \bar{\partial}_k \tilde{f}(\mathbf{z}) = i\int_{\mathbb{R}^n} d^n\mathbf{p}\; (\mathbf{p}\cdot\mathbf{y})\, \delta(\mathbf{p}\cdot\mathbf{y})\, e^{2\pi i \mathbf{p}\cdot\mathbf{x}}\, \hat{f}(\mathbf{p}) = 0. \tag{2.8}$$

Thus $\tilde{f}(\mathbf{z})$ *is analytic in the direction* $\mathbf{y}$. In the one–dimensional case, this reduces to

$$\frac{\partial \tilde{f}(z)}{\partial z^*} = 0 \quad \forall y \neq 0, \tag{2.9}$$

which states that $\tilde{f}(z)$ is analytic in the upper– and lower–half planes. In one dimension, there are only *two* imaginary directions, whereas in n dimensions, every $\mathbf{y} \neq \mathbf{0}$ defines an imaginary direction.

The multivariate AST is related to the *Hilbert transform in the direction* $\mathbf{y}$ (Stein [1970], p. 49), defined as

$$H_{\mathbf{y}} f(\mathbf{x}) = \frac{1}{\pi} \mathrm{PV} \int_{-\infty}^{\infty} \frac{du}{u} f(\mathbf{x} - u\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \ \mathbf{y} \neq \mathbf{0}. \tag{2.10}$$

(Usually, it is assumed that $\mathbf{y}$ is a unit vector; we do not make this assumption.) Namely, an argument similar to the above shows that

$$f(\mathbf{x}) \pm i H_{\mathbf{y}} f(\mathbf{x}) = 2 \lim_{\epsilon \downarrow 0} \tilde{f}(\mathbf{x} \pm i\epsilon\mathbf{y}), \tag{2.11}$$

hence

$$H_{\mathbf{y}} f(\mathbf{x}) = \frac{1}{i} \lim_{\epsilon \downarrow 0} [\tilde{f}(\mathbf{x} + i\epsilon\mathbf{y}) - \tilde{f}(\mathbf{x} - i\epsilon\mathbf{y})], \tag{2.12}$$

which, for $n = 1$ and $y > 0$, reduces to the previous relation with the ordinary Hilbert transform.

As in the one–dimensional case, $f(\mathbf{x})$ is the boundary–value of $\tilde{f}(\mathbf{z})$ in the sense that

$$f(\mathbf{x}) = \lim_{\epsilon \to 0} [\tilde{f}(\mathbf{x} + i\epsilon\mathbf{y}) + \tilde{f}(\mathbf{x} - i\epsilon\mathbf{y})]. \tag{2.13}$$

For real–valued f, these equations reduce to

$$\begin{aligned} f(\mathbf{x}) &= 2 \lim_{\epsilon \to 0} \Re \tilde{f}(\mathbf{x} + i\epsilon\mathbf{y}) \\ H_{\mathbf{y}} f(x) &= 2 \lim_{\epsilon \downarrow 0} \Im \tilde{f}(\mathbf{x} + i\epsilon\mathbf{y}). \end{aligned} \tag{2.14}$$

3. Applications

The AST originated in the study of quantized fields (Kaiser [1987]) and is an example of a *Windowed X–Ray transform* (WXT), defined and studied in Kaiser [1990a,b]. In the terminology introduced there, it corresponds to the window function

$$h(\tau)^* = \frac{1}{2\pi i(\tau - i)}. \tag{3.1}$$

The WXT, in turn, is a multidimensional generalization of the wavelet transform (Daubechies [1988], Meyer [1990]). The window function in eq. (3.1) is not "admissible" in the sense of affine wavelets, hence the general reconstruction formula developed for the WXT fails. However, there are alternative ways to recover the original signal.

Example 1: Hardy spaces

Suppose that $\hat{f}(\mathbf{p})$ vanishes outside of some closed convex cone V_+. The cone V'_+ *dual* to V_+ is defined as

$$V'_+ = \{\mathbf{y} \in \mathbb{R}^n \,|\, \mathbf{p} \cdot \mathbf{y} > 0 \ \forall \mathbf{p} \in V_+\}, \tag{3.2}$$

and it is clearly an open convex cone. Note that for $\mathbf{y} \in V'_+$, $\theta(\mathbf{p} \cdot \mathbf{y}) \equiv 1$ on the support of $\hat{f}$ (except at $\mathbf{p} = \mathbf{0}$, which has measure 0), hence if $f \in L^2(\mathbb{R}^n)$,

$$\tilde{f}(\mathbf{z}) = \int_{V_+} d^n\mathbf{p}\, e^{2\pi i \mathbf{p} \cdot (\mathbf{x} + i\mathbf{y})}\, \hat{f}(\mathbf{p}) \tag{3.3}$$

and it follows (Stein and Weiss [1971]) that $\tilde{f}(\mathbf{z})$ is analytic in the *tube domain*

$$\mathcal{T}_+ \equiv \{\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n \,|\, \mathbf{y} \in V'_+\}. \tag{3.4}$$

The set $H^2 \equiv \{\tilde{f} \,|\, f \in L^2(V_+)\}$ is known as a *Hardy space.* Note also that $\tilde{f}$ vanishes in the tube

$$\mathcal{T}_- \equiv \{\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n \,|\, -\mathbf{y} \in V'_+\}, \tag{3.5}$$

since there $\mathbf{p} \cdot \mathbf{y} < 0$ for all $\mathbf{0} \neq \mathbf{p} \in V_+$. Thus eq. (2.13) gives

$$f(\mathbf{x}) = \lim_{\epsilon \downarrow 0} \tilde{f}(\mathbf{x} + i\epsilon\mathbf{y}), \qquad \mathbf{y} \in V', \tag{3.6}$$

which states that f is a *boundary–value* of $\tilde{f}$. Since $\tilde{f}(\mathbf{z})$ is analytic, it may be regarded as a *regularization* of $f(\mathbf{x})$ (the latter, being merely square–integrable, is a distribution). Eq. (3.6) can be viewed as a "reconstruction" of f from $\tilde{f}$, albeit a somewhat trivial one.

Example 2: The Klein–Gordon Equation

An important application of the AST is to signals which satisfy some partial differential equations. (In fact, it was in this context that the transform originated.) Suppose that f satisfies the *Klein–Gordon equation* in $\mathbb{R}^n$,

$$\Box\, f + m^2 c^4 f = 0, \tag{3.7}$$

where

$$\Box \equiv \frac{\partial^2}{\partial x_1^2} - c^2 \frac{\partial^2}{\partial x_2^2} - \cdots - c^2 \frac{\partial^2}{\partial x_n^2} \tag{3.8}$$

is the D'Alembertian or wave operator for waves with propagation speed c. Here $\mathbb{R}^n$ is interpreted as *space–time,* with x_1 the time coordinate and $(x_2, \cdots, x_n)$ the space coordinates, and $m > 0$ is a mass parameter. This equation describes relativistic particles in quantum mechanics. The limit $m \to 0$ gives the wave equation, which will be discussed below. Define the *solid light cone* in Fourier space by

$$V = \{\mathbf{p} \in \mathbb{R}^n \,|\, \mathbf{p}^2 \equiv p_1^2 - c^2 p_2^2 - \cdots c^2 p_n^2 \geq 0\}. \tag{3.9}$$

(Note that we are now using a Lorentz metric.) V is the union of the *forward* and *backward* light cones V_+ and V_-, where $p_1 \geq 0$ and $p_1 \leq 0$, respectively. Note that $V_\pm$ are convex but V is not. The fact that f satisfies the Klein–Gordon equation means that its Fourier transform $\hat{f}$ is supported on the double mass hyperboloid

$$\Omega_m = \{\mathbf{p} \in \mathbb{R}^n \,|\, \mathbf{p}^2 = m^2 c^4\} = \Omega_m^+ \cup \Omega_m^-, \tag{3.10}$$

where $\Omega_m^\pm \subset V_\pm$. Thus $\hat{f} = \hat{f}_+ + \hat{f}_-$, where $\hat{f}_\pm$ are distributions supported on $\Omega_m^\pm$. Since $\Omega_m^\pm \subset V_\pm$, an argument similar to that used for Hardy spaces shows that the corresponding solutions $f_\pm(\mathbf{x})$ have AST's $\tilde{f}_\pm(\mathbf{z})$ which are analytic in $\mathcal{T}_\pm$ and vanish in $\mathcal{T}_\mp$, where

$$\mathcal{T}_\pm = \{\mathbf{x} + i\mathbf{y} \in \mathbb{C}^n \,|\, \mathbf{y} \in V'_\pm\} \tag{3.11}$$

and $V'_\pm$ are the cones dual to $V_\pm$, which can be seen to be

$$V'_\pm \equiv \{\mathbf{y} \in \mathbb{R}^n \,|\, \mathbf{y}^2 \equiv c^2 y_1^2 - y_2^2 - \cdots - y_n^2 > 0, \quad \pm y_1 > 0\}. \tag{3.12}$$

Note that while V is a cone in *Fourier space* (i.e., p_1 is a frequency and $p_2, \cdots p_n$ are wave numbers per unit length), $V' \equiv V'_+ \cup V'_-$ is a cone in space–time. Technically, these two spaces are dual and should not be identified with one another.

The AST of f, $\tilde{f}(\mathbf{z}) = \tilde{f}_+(\mathbf{z}) + \tilde{f}_-(\mathbf{z})$, is therefore analytic in the *double tube* $\mathcal{T} \equiv \mathcal{T}_+ \cup \mathcal{T}_-$, with $\mathcal{T}_+$ and $\mathcal{T}_-$ containing only the positive– and negative–frequency parts of f, respectively. This "polarization" of frequencies is important because it makes it possible to reconstruct the solution f from $\tilde{f}$ without approaching the singular boundary $\mathbb{R}^n$ ($\mathbf{y} \to \mathbf{0}$). Eq. (3.3) shows that $\tilde{f}_\pm$ have the form

$$\tilde{f}_\pm(\mathbf{z}) = \int_{\Omega_m^\pm} d\tilde{p}\ \hat{e}_{\mathbf{z}}^\pm(\mathbf{p})^*\, a_\pm(\mathbf{p}), \qquad \mathbf{z} \in \mathcal{T}_\pm, \tag{3.13}$$

where $d\tilde{p}$ is the induced measure on Ω_m and

$$\hat{e}_{\mathbf{z}}^\pm(\mathbf{p})^* \equiv e^{2\pi i \mathbf{p}\cdot\mathbf{z}}, \qquad \mathbf{z} \in \mathcal{T}_\pm,\ \mathbf{p} \in \Omega_m^\pm. \tag{3.14}$$

The corresponding expression $e_{\mathbf{z}}^\pm$ in the space–time domain, defined by

$$e_{\mathbf{z}}^\pm(\mathbf{x}')^* = \int_{\Omega_m^\pm} d\tilde{p}\ e^{2\pi i \mathbf{p}\cdot(\mathbf{z}-\mathbf{x}')}, \tag{3.15}$$

is a solution of the wave equation which can be shown to be a *coherent wave–packet* whose parameters $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ have a direct geometric interpretation: $\mathbf{x}$ is a point in space–time about which $e_{\mathbf{z}}^\pm$ is "focused" (i.e., $e_{\mathbf{z}}^\pm(\mathbf{x}')$ converges toward the point $(x_2, \cdots, x_n)$ for times $x_1' < x_1$ and diverges away from it for times $x_1' > x_1$), and $\mathbf{y}$ is a set of homogeneous coordinates for the *average velocity* at which $e_{\mathbf{z}}^\pm$ is traveling. Furthermore, the invariant $\lambda > 0$ defined by $\lambda^2 \equiv \mathbf{y}^2$ can be interpreted as a *scale parameter* which, roughly speaking, measures the sperad (resolution) of the wave packet at the instant of its maximal focus ($x_1' = x_1$) in its rest–frame. The positive– and negative–frequency packets $e_{\mathbf{z}}^+$ and $e_{\mathbf{z}}^-$ are interpreted in quantum theory as particles and antiparticles, respectively. (This agrees with the usual observation that antiparticles "go backward in time.") See Kaiser [1977, 1990a] for details.

Since the window function $h(\tau)$ used in the AST has Fourier transform $\hat{h}(\xi) = \theta(\xi)\, e^{-2\pi\xi}$, it is not admissible in the general sense developed in Kaiser [1990b], i.e.

$$\int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\hat{h}(\xi)|^2 = \infty. \tag{3.16}$$

That admissibility condition was associated with a reconstruction formula which represents f as an integral of generalized wavelets parametrized by *all* of $\mathbb{R}^n \times \mathbb{R}^n_*$, i.e. all $\mathbf{x} + i\mathbf{y}$ with $\mathbf{y} \neq \mathbf{0}$. This was acceptable when considering general functions $f(\mathbf{x})$ in $L^2(\mathbb{R}^n)$, since then we could define a representation of the affine group on such functions. But now we are dealing with a Hilbert space $\mathcal{H}$ of solutions of the Klein–Gordon equation,

$$f(\mathbf{x}) = \int_{\Omega_m} d\tilde{p}\, e^{2\pi i \mathbf{p}\cdot\mathbf{x}}\, a(\mathbf{p}), \tag{3.17}$$

with norm

$$\|f\|^2 \equiv \int_{\Omega_m} d\tilde{p}\, |a(\mathbf{p})|^2, \tag{3.18}$$

and general affine transformations no longer map solutions to solutions, i.e. they no longer define operators on $\mathcal{H}$. The mass m spoils the invariance of the equation under dilations. Only the subgroup $\mathcal{P}$ of translations together with Lorentz transformations (i.e., linear maps $\mathbf{y} \mapsto A\mathbf{y}$ which preserve the Lorentz norm $\mathbf{y}^2$) map solutions to solutions. $\mathcal{P}$ is called the inhomogeneous Lorentz or *Poincaré* group. Recall that the measure used in the reconstruction formula was chosen to be invariant under dilations and rotations. Since dilations no longer define operators on $\mathcal{H}$, this measure is no longer appropriate. Rather, we now expect to reconstruct f by integrating in $\mathbf{y}$ over the double hyperboloid

$$\Omega_\lambda = \{\mathbf{y} \in V' \,|\, \mathbf{y}^2 = \lambda^2\} = \Omega_\lambda^+ \cup \Omega_\lambda^- \tag{3.19}$$

for an arbitrary fixed $\lambda > 0$. Furthermore, we do not expect to integrate over all $\mathbf{x} \in \mathbb{R}^n$, since a solution is determined by its data on any *Cauchy surface* $S \subset \mathbb{R}^n$. For simplicity, take S to be the hyperplane $x_1 = t$ for fixed $t \in \mathbb{R}$, though any Cauchy surface (spacelike $(n-1)$–dimensional submanifold of $\mathbb{R}^n$) will do. Thus consider the $(2n-2)$–dimensional submanifold

$$\sigma = \{\mathbf{x} + i\mathbf{y} \in \mathcal{T} \,|\, x_1 = t,\ \mathbf{y} \in \Omega_\lambda\} = \sigma_+ \cup \sigma_-, \tag{3.20}$$

where $\mathbf{y} \in \Omega_\lambda^\pm$ in $\sigma_\pm$. σ parametrizes all possible locations and velocities of a classical particle at the fixed time t, i.e. it is a *phase space*. A reconstruction formula has been obtained in the form

$$f(\mathbf{x}') = \int_\sigma d\mu(\mathbf{z})\, e_{\mathbf{z}}(\mathbf{x}')\, \tilde{f}(\mathbf{z}), \tag{3.21}$$

where $e_{\mathbf{z}} \equiv e_{\mathbf{z}}^\pm$ on $\sigma_\pm$, σ is parametrized by $(x_2, \cdots x_n, y_2, \cdots y_n)$, and

$$d\mu(\mathbf{z}) = A(\lambda, m)^{-1} dx_2 \cdots dx_n dy_2 \cdots dy_n. \tag{3.22}$$

$A(\lambda, m)$ is a certain constant related to the admissibility of the basic wavelet h with respect to the measure $dx_2 \cdots dy_n$. Note that this differs from the usual construction of a solution from its initial data, which uses the values of both f and $\partial f/\partial x_1$ on S. The intuitive explanation is that the dependence of $\tilde{f}(\mathbf{x}+i\mathbf{y})$ on $\mathbf{y}\in\Omega_\lambda$, for fixed $\mathbf{x}\in S$, gives the equivalent "velocity" information. The independence of the reconstruction from the choice of Cauchy surface is due to a conservation law satisfied by solutions.

The above reconstruction formula bears a close resemblance to the standard representation of a function in terms of wavelets, for the following reason: In the hyperbolic geometry of spacetime, a moving object undergoes a *Lorentz contraction,* i.e. it shrinks in its direction of motion. Since $\mathbf{y}$ represents a velocity, eq. (3.21) expresses f as a linear combination of "wavelets" centered about all possible points in space (at time t) and in various states of compression. However, the analogy is incomplete since the $e_{\mathbf{z}}$'s can only contract and not dilate. (That is, they have a minimum width in their rest frames.) Their contraction is due to Lorentz transformations rather than ordinary dilations of the form $\mathbf{x}\mapsto a\mathbf{x}$, $a\neq 0$. As noted earlier, the Klein–Gordon equation is not invariant under such dilations, due to the presence of $m>0$. On the other hand, the wave equation ($m\to 0$) *is* invariant under dilations, hence the analogy with wavelets can be expected to be closer.

Example 3: Wavelets and The Wave Equation?

A reconstruction formula for solutions of the wave equation similar to eq. (3.21) should be of great interest to signal analysis, for the following reason: Many signals naturally ocurring in Nature, such as sound waves, electric fileds and magnetic fields, satisfy the wave equation away from sources. A representation of such signals as linear combinations of coherent wave packets, parametrized by location and velocity, would give a kind of "geometrical optics" picture of their content. Unfortunately, the reconstruction formula in eq. (3.21) fails when $m\to 0$ because $A(\lambda,m)$ diverges as $m\to 0$. That is, the basic wavelet $h(\tau)$ is no longer admissible with respect to $dx_2\cdots dy_n$ when $m=0$. The reason for this is probably related to the fact that this formula was derived in the context of the Poincaré group, which is the invariance group of the Klein–Gordon equation, and the invariance group of the wave equation is the *conformal group* $\mathcal{C}$, which contains the Poincaré group as well as dilations and accelerations. It may be that once this larger group is taken into account, an appropriate reconstruction formula can be found.

Remark: The wavelet analysis developed above for the Klein–Gordon equation has the interesting feature that the wavelets are "dedicated" to the equation rather than being merely a convenient set of functions to be used in expansions. (This is somewhat reminiscent of the situation in the spectral theorem, where expansions are customized to a given operator.) The reward for this naturality is that symmetry operations (such as translations, rotations and Lorentz transformations) take wavelets to wavelets. This can be expected to have the practical consequence of making the description economical and precise.

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